"Approaching sing theory via derived categorical techniques"

<u>The gist</u>: der. cats + sing. theory

why important in modern AG/CA?

↳ different flavors, some more natural via homo. methods

rat, DB, der. spl.

- Generation for Δ-cats
- Recent development via gen. techniques
- Future directions (if time permits)

<u>Def</u>: A Δ-cat is an

- (additive) cat. $\mathcal{J}$
- autoequivalence $[1] : \mathcal{J} \to \mathcal{J}$ ← called shift
- collection of diagrams

$$A \to B \to C \to A[1]$$

called dist Δ's

Satisfying certain axioms.

Def: $\mathcal{T}$ Δ-cat

$f: A \to B$

A <u>cone</u> of f, denoted cone(f), is any object C fitting into a dist. Δ-

$$A \to B \to C \to A[1].$$

Def: $\mathcal{T}$ Δ-cat, $\mathcal{C} \subseteq \mathcal{T}$ subcat

- $\mathcal{C}$ is <u>strictly full</u> if it is closed under iso's & is a full subcat,
- $\mathcal{C}$ is <u>Δ-subcat</u> if it is st. full and closed under shifts + cones (i.e. $A \to B \to C \to A[1]$ w/ $A, C \in \mathcal{C}$ $\Rightarrow B \in \mathcal{C}$)
- $\mathcal{C}$ is <u>thick</u> if it is a Δ-subcat closed under dir. summands.
- $\langle \mathcal{C} \rangle$ denotes the smallest thick subcat of $\mathcal{T}$ containing $\mathcal{C}$ (always exist via $\bigcap$)

Q: X a Noetherian scheme (or ring!)

$\mathrm{coh}(X)$ ab. cat. coh. sheaves

- $D^b_{coh}(X)$ bnd der cat of coh

Here objects are of the form

$$0 \to E_s \xrightarrow{d_s} E_{s-1} \xrightarrow{d_{s-1}} \cdots \xrightarrow{d_{s+a}} E_{s+a+1} \to 0$$

s.t. $d_i \circ d_{i-1} = 0$ and $E_i \in \mathrm{coh}(X)\ \forall i$

- $\mathrm{Perf}(X) \subseteq D^b_{coh}(X)$ consists of "perfect cmplx"

There are objects affine locally of the form

$$0 \to \mathcal{O}_U^{\oplus r_s} \to \mathcal{O}_U^{\oplus r_{s-1}} \to \cdots \to \mathcal{O}_U^{\oplus r_{s+a}} \to 0$$

where $r_i \geq 0$. This is a thick subcat.

Th: X is regular $\Longleftrightarrow \mathrm{Perf}(X) = D^b_{coh}(X)$

↳ This takes a lot of work to show

Reduce to local case

Use Auslander–Buchsbaum–Serre

Def: $\mathcal{T}$ Δ-cat., $\mathcal{C} \subseteq \mathcal{T}$

- add($\mathcal{S}$) smallest st. full subcat of $\mathcal{T}$ closed under $[1]$, finite coproducts, and dir. summands

- Inductively define

$$\langle \mathcal{S} \rangle_n := \begin{cases} \text{add}(0) & , n = 0 \\ \text{add}(\mathcal{S}) & , n = 1 \\ \text{add}\left(\{\text{cone}(\phi) : \phi \in \text{Hom}(\langle \mathcal{S} \rangle_{n-1}, \langle \mathcal{S} \rangle_1)\}\right) & , n > 1. \end{cases}$$

called 'n^{th} thickening'

Rmk: $\langle \mathcal{S} \rangle = \bigcup_{n=0}^{\infty} \langle \mathcal{S} \rangle_n$

↳ gives an exhaustive filtration

Def: $\mathcal{T}$ Δ-cat, $G \in \mathcal{T}$ is called

- <u>classical gen.</u> if $\langle G \rangle = \mathcal{T}$
- <u>strong gen.</u> if $\exists n \geq 0$ s.t. $\langle G \rangle_n = \mathcal{T}$

↳ gives a finer way to study objects in $\mathcal{T}$ by working w/ single object

↳ Classical and strong gen. play an important role in alg + geo (see later)

↳ Abstractly, they imply many "representability" results

Ex: • $D^b_{coh}(X)$ admits st. gen. if $\xrightarrow{\text{use Gabber's result wry.}}$
X qc sep. quasi-exc. dim X (Aoki)

• $D^b_{coh}(X)$ admits classical gen. if
e.g. closed subscheme of X
has open regular locus (DL)
↳ cf. IT, ELS (J-2)

• $\mathrm{Perf}(X)$ admits a classical gen.
$\forall$ qc qc X (Bondal - Van den Bergh)

At times we can be explicit:

• X sm. quasi-proj $/k$, $\mathcal{L}$ ample
$\bigoplus_{i=0}^{\dim X} \mathcal{L}^{\otimes i}$ st. gen. $D^b_{coh}(X)$ $(= \mathrm{Perf}(X))$
(Rouq., Orlov, BVdB)

- X quasi-proj var. $/k$, L ample

$\mathrm{char}(k) > 0 \implies F^e_*\left(\bigoplus_{i=0}^{\dim X} L^{\otimes i}\right)$ st. gen.

for $D^b_{coh}(X)$ $\forall e >> 0$ (BILMP)

↳ ∃ bound on e in terms of codepth and char(p)

- X var. over fld k

$f : \hat{X} \to X$ modif. w/ $\hat{X}$ smooth $/k$

∃ L ample on $\hat{X}$

$\implies \mathbb{R} f_*\left(\bigoplus_{i=0}^{\dim X} L^{\otimes i}\right)$ st. gen for $D^b_{coh}(X)$ (L.)

<u>Th</u>: (EL) X connected Noeth. affine scheme

$\mathcal{G} \subseteq D^b_{coh}(X)$ st. $\mathcal{G} = \langle G \rangle_n$

$\implies \mathcal{G} = D^b_{coh}(X)$ or $\mathcal{G} = 0$.

↳ ∃ global cases (i.e. non-affine)

where this is false (e.g. SOD's

of proj. bundles w/ $D^b_{coh}(X) = \langle G \rangle_n$)

see BS 'conv.'

Rmk: $\mathcal{T}$ Δ-cat

$G, E \in \mathcal{T}$ s.t. $E \in \langle G \rangle$

$\Rightarrow$ $\exists\, n \geq 0$ s.t. $E \in \langle G \rangle_n$

b/c $\langle G \rangle = \bigcup_{n=0}^{\infty} \langle G \rangle_n$

↳ asking for smallest number of cones...

Def: (ABIM, R) $\mathcal{T}$, Δ-cat; $S \subseteq \mathcal{T}$; $E, G \in \mathcal{T}$

- The level of E w.r.t. S is

$$\mathrm{level}^{S}(E) := \inf\{n \geq 0 : E \in \langle S \rangle_n\}$$

↳ set ∞ if no such n exists

- G classical gen. for $\mathcal{T}$,

$$\text{gen.time}(G) := \sup_{E \in \mathcal{T}} \{\mathrm{level}^{G}(E)\} - 1$$

↳ keeps track of cones

↳ —//—

- The (Rouq.) dim of $\mathcal{T}$ is

$$\dim \mathcal{T} := \inf\{\text{gentime}(G) : G \text{ cl. gen. } \mathcal{T}\}$$

Lem: (ABIM) $\mathcal{G}$ Δ-cat, $\mathcal{S} \subseteq \mathcal{G}$

- $\mathrm{lvl}^{\mathcal{S}}(E) = \mathrm{lvl}^{\mathcal{S}}(E[n])$ $\forall n \in \mathbb{Z}$ $\forall E \in \mathcal{G}$
- $\mathrm{lvl}^{\mathcal{S}}(B) \leq \mathrm{lvl}^{\mathcal{S}}(A) + \mathrm{lvl}^{\mathcal{S}}(C)$ $\forall \Delta$- $A \to B \to C \rightsquigarrow$
- $\mathrm{lvl}^{\mathcal{S}}(A \oplus B) = \sup\{\mathrm{lvl}^{\mathcal{S}}(A), \mathrm{lvl}^{\mathcal{S}}(B)\}$ $\forall A, B \in \mathcal{G}$
- $F : \mathcal{G} \to \mathcal{G}'$ Δ-functor, $\mathrm{lvl}^{F(\mathcal{S})}(F(E)) \geq \mathrm{lvl}^{\mathcal{S}}(E)$ $\forall E \in \mathcal{G}$

  $\hookrightarrow$ exact

Lem: (Lete) X affine Noeth. sch., $E, G \in D^b_{coh}(X)$

$$\mathrm{lvl}^{G}(E) = \sup_{p \in X}\{\mathrm{lvl}^{G_p}(E_p)\}$$

Ex: Fails for non-affine, e.g.

$\mathbb{P}^1_{\mathbb{C}}$ w/ $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ locally at stalks,

yet $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1) \notin \langle \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \rangle$ b/c

$$\mathrm{Ext}^0(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = \mathbb{C}$$

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = 0$$

Rmk: X Noeth.

$\overset{L}{\otimes} : \mathrm{Perf}(X) \times D^b_{coh}(X) \to D^b_{coh}(X)$ tensor act.

à la Stevenson

Lem: (BILMP)

X Noeth sch.

$E, G \in D^b_{coh}(X)$; $\langle P \rangle = \mathrm{Perf}(X)$

$E_p \in \langle G_p \rangle \ \forall p \in X \Rightarrow E \in \langle P \overset{L}{\otimes} G \rangle$

↳ allows to leverage local-to-global information for generation

Aside: A conjecture of Orlov states

$\forall$ sm. var. X / k (any char), one has

$\dim D^b_{coh}(X) = \dim X$

- $\dim X = 1$
- various homog spaces
- various Fano, CY var.
- toric var (using HMS)

($\Rightarrow$'s same for wk. sing. varieties)

Ex: R reg. local ring

↳ walk through pdim to all replace to show $\langle R\rangle_{\dim R+1} = D^b_{coh}(R)$

↳ mention Lutz recent

$\Rightarrow$ reg. quasi-affine sch $\dim X$ are those $\dim D^b_{coh}(X) = \dim X$.

Q: Can we use generation tech. to study the sing. of a Noeth sch.?

↳ what does this mean?

↳ Can we characterize?

Lem: (L.) X var. / $\mathbb{C}$

$f: \tilde{X} \to X$ modif. w/ $\tilde{X}$ sm.

$\exists N \geq 0$ s.t. $\langle \mathbb{R} f_* D^b_{coh}(\tilde{X}) \rangle_N = D^b_{coh}(X)$

Ex: Suppose X has rational sing., i.e.

$$\mathcal{O}_X \xrightarrow{\text{ntrl}} \mathbb{R} f_* \mathcal{O}_{\tilde{X}} \quad \text{is an iso. in } D^b_{Coh}(X)$$

Kovács $\Rightarrow$ X has rat. sing $\Leftrightarrow$ $\mathcal{O}_X \xrightarrow{\text{ntrl}} \mathbb{R} f_* \mathcal{O}_{\tilde{X}}$ splits

A conseq. to above in $N=1$.

We can ask the converse to example above.

Thm: (L., Venkatesh) X var. / $\mathbb{C}$

X has rat. sing. $\Leftrightarrow$ $\mathcal{O}_X \in \langle \mathbb{R} f_* D^b_{Coh}(\tilde{X}) \rangle_1$

$\Leftrightarrow$ $D^b_{Coh}(X) = \langle \mathbb{R} f_* D^b_{Coh}(\tilde{X}) \rangle_1$

where $f: \tilde{X} \to X$ is modif. from smooth var.

Rmk: $\exists\, N \gg 0$ s.t. $D^b_{Coh}(X) = \langle \mathbb{R} f_* D^b_{Coh}(\tilde{X}) \rangle_N$

$\therefore$ N "measures" failure of rat. sing.

$\hookrightarrow$ forthcoming work w/ De Deyn, Manali-Rahul & Venkatesh

Q: Can we bound N?

Ex: (L., Venkatesh)

① X quasi-proj curve $/ \mathbb{C}$

$f: \tilde{X} \to X$ be mod. w/ $\tilde{X}$ sm. (i.e. normal.)

$\therefore N \leq 1 + \sup_{p \in \mathrm{Sing}(X)} \{\delta_s\}$ where

δ_s is the δ-inv. of $s \in X$

(i.e. $\mathrm{length}_{\mathcal{O}_{X,p}} (A_p / \mathcal{O}_{X,p})$ w/ A_p int. closure in $k(p)$)

② $X \subseteq_{cl.} \mathbb{P}^n_{\mathbb{C}}$ smooth hypersurface of degree $d \geq n$

C = affine cone of X associated to ample line on X

$\therefore N \leq 1 + 2(d-n)$

↳ further w/ DMNLV, we can bound var. w/ iso. sing.

Now we turn gears to prime char.

Ex: • R Noeth of prime char p

$\exists$ "Frob. morphism" $F_*: R \circlearrowleft \quad r \longmapsto r^p$

$F_*^e := \underbrace{F_* \circ \cdots \circ F_*}_{e \text{ times}}$

• We say R is "F-finite" if F_* is finite

• We say F-finite R is "F-split" if $R \xrightarrow{\text{ntrl}} F_*R$ splits

• BILMP showed $\langle F_*^e R \rangle_N = D^b_{coh}(R) \ \forall e \gg 0$

where R F-finite

Q: $N=1 \longleftrightarrow R$ is F-split?

A: yes! (think why ....)

Ex (BILMP): R is an F-finite C.I. of char p

(i.e. $R \cong A/I$ where I f.g. by regular seq. in reg.)

$\text{codepth}(R) = \text{edim}(R) - \text{depth}(R)$

$N \leq p^{\text{codepth}(R)}$

↳ Extends to LCI's, no non-local rings

Q: So we have seen generation can characterize and measure failure of certain sing. Can we use it to define a new class of sing?

Rmk: X var $/\mathbb{C}$ w/ rational sing
$f: \hat{X} \to X$ modif. by sm. var.

$\Rightarrow \mathcal{O}_X \xrightarrow{\text{q-iso}} \mathbb{R}f_*\mathcal{O}_{\hat{X}}$, so

$$\begin{aligned} H^n(\hat{X}, \mathcal{O}_{\hat{X}}) &= \mathrm{Ext}^n(\mathcal{O}_{\hat{X}}, \mathcal{O}_{\hat{X}}) \\ &= \mathrm{Ext}^n(Lf^*\mathcal{O}_X, \mathcal{O}_{\hat{X}}) \\ &= \mathrm{Ext}^n(\mathcal{O}_X, \mathbb{R}f_*\mathcal{O}_{\hat{X}}) \\ &= \mathrm{Ext}^n(\mathcal{O}_X, \mathcal{O}_X) = H^n(X, \mathcal{O}_X) \end{aligned}$$

$\therefore$ $\underbrace{\text{sheaf coh.}}$ of X sim. to $\hat{X}$

↳ very classical measure of the hom. alg. info on X

Rmk: • k perfect fld of char $p > 0$

- X k proj var
- L ample line bundle on X
- BILMP $\Rightarrow$ $F^e_*\left(\bigoplus_{i=0}^{\dim X} L^{\otimes i}\right)$ st. gen

Q: What happens if $F^e_* \mathcal{O}_X$ st. gen.?

- This is always the case for X quasi-affine
- So we confine to projective case

Ex: k alg. closed char $p>0$

$$D^b_{coh}(\mathbb{P}^n) = \left\langle \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(-i) \right\rangle_N$$

↳ Beilinson

$$F^e_* \mathcal{O}_{\mathbb{P}^n} \cong \bigoplus_{i\geq 0} \mathcal{O}_X(-i)^{\oplus \alpha(i,e)}$$

where $\alpha(i,e)$ = # of monomials of degree ip not divisible by any p-th power of a variable

↳ Rao (if not mistaken)

$\therefore$ $\forall e \gg 0$, $\langle F^e_* \mathcal{O}_{\mathbb{P}^n} \rangle = D^b_{coh}(\mathbb{P}^n)$

Def: X Noeth. F-fin.

X is 'F-thick' if $\langle F^e_* \mathcal{O}_X \rangle = D^b_{coh}(X)$ $\forall e \gg 0$

Q: What does F-thickness tell us?

What examples or counter examples are there?

↳ $\mathbb{P}^n$ + quasi-affine so far!

Ex: (toric varieties)

Hanlon-Hicks-Lazarev, Erman-Brown, Favero-Huang

$\Rightarrow$ any toric variety is F-thick

Ex: Blow up $\mathbb{P}^2$ at 4 or less pts gen. pos.

$\Rightarrow$ is F-thick (Hara)

Ex: $X \subseteq \mathbb{P}^{n+1}_k$ sm. quadric w/ $k = \bar{k}$, ~~p~~ char $k > 0$

X F-thick if n even & $n \geq 2(p+1)$

or n odd & $n \geq 3p+2$

(Samokhin)

Ex: (BILMP) $k = \bar{k}$, $\operatorname{char}(k) = p > 0$

C smooth proj. curves $/k$

C F-thick $\iff$ $\operatorname{genus}(C) = 0$

↳ Use semistability of vector bundles

Ex: (BILMP)

$\pi : X \to C$ sm. proj ruled surface

C sm proj curve w/ genus > 0

$k = \bar{k}$, $\operatorname{char}(k) = p > 0$

$\Rightarrow X$ not F-thick

↳ Play a game of SOD's and use curves classif.

Ex: (BILMP) Abelian var over alg closed pos char are not F-thick

Thm: (BILMP)

X is F-thick $\Longrightarrow$ $\exists\, n \geq 0$ s.t.

$$H^*(X, \mathcal{L}^{\otimes n}) \neq 0 \quad \forall\, \mathcal{L} \in \mathrm{Pic}(X)$$

Q: Can we find more examples or counterexamples of F-thick? What properties do they satisfy?

↳ Malloy has recent work on the "tilting" case for del Pezzo's

↳ F-thickness is a sort of "categorical singularity"

Q: • Do these techniques fit into a bigger picture?

↳ so far, characterize regularity

- rat'l sing in char zero
- F-split in char p > 0
- F-thickness

⌈measuring⌋ + ⌈homolog. flavors to sing⌋

Def: X int. Noeth.

- X has 'rat. sing' if $\exists\, f: \tilde{X} \to X$ modif from reg sch s.t. $\mathcal{O}_X \underset{ntrl}{\xrightarrow{\sim}} Rf_*\mathcal{O}_{\tilde{X}}$
- X has '<u>mild</u> rat sing' if $\exists\, f: \tilde{X} \to X$ modif frm reg sch. s.t. $\mathcal{O}_X \xrightarrow{ntrl.} Rf_*\mathcal{O}_{\tilde{X}}$ splits
- (Bhatt) X is a 'der. spl' if $\mathcal{O}_X \xrightarrow{ntrl} Rf_*\mathcal{O}_Y$ splits $\forall\, f: Y \to X$ proper surj.

unpublished

- (Kovacs) X is a 'br. der. spl' if $\mathcal{O}_X \xrightarrow{ntrl} Rf_*\mathcal{O}_Y$ splits $\forall\, f: Y \to X$ modif.

Rmk:

$$\begin{tikzcd}
\text{br. der. spl.} & & \text{rat sing} \arrow[ll, bend right, "\text{affine q.e. } \mathbb{Q}\text{-sch. hyp}"'] \arrow[dl, "\text{q.c. q.e. } \mathbb{Q}\text{-sch. Murayama}"'] \arrow[dr] & \\
& \text{der. spl.} & & \text{mild rat sing}
\end{tikzcd}$$

- In chap p: Elliptic is bds but not ds

- $\mathbb{Q}$-ech. case relies on GR-vanishing

Lem: X int Noeth. TFAE

- X bds
- $\mathcal{O}_X \xrightarrow{\text{ntrl}} \mathbb{R}f_*\mathcal{O}_{X'}$ splits $\forall$ blowup $X' \xrightarrow{f} X$

↳ along nonzero ideal sheaf

Thm X int Noeth

$\exists$ $f: \widetilde{X} \to X$ modif from reg.

TFAE

- X mild rat sing
- X bsd

↳ proof uses no GR-vanishing, duality, holds in char.

Cor: X q.c. q.e. int. $\mathbb{Q}$-sch. TFAE

- rat sing
- mild rat sing
- der spl
- bir der spl.

} captures the story (expected) in $\mathbb{Q}$-char