CHATZISTAMATIOU–RÜLLING HIGHER VANISHING IN THE NON-EXCELLENT CASE

PAT LANK

ABSTRACT. This is an expository note on folklore regarding the vanishing of higher cohomology for the structure sheaf under projective birational morphisms of quasi-compact regular schemes. A result of Chatzistamatiou–Rülling can be strengthened by removing excellence and the existence of dualizing complex constraints. The simple recipe to do so uses classical geometry to reduce the problem to local algebra. Assuming a conjecture of Grothendieck, we explain how to generalize to proper morphisms.

Lemma 0.1. *Let $f\colon Y \to X$ be a morphism of Noetherian scheme. Suppose (P) is a property of Noetherian rings that is stable under under localization (e.g. integral, regular, etc). If Y satisfies (P) affine locally (i.e. admits affine open cover whose components satisfy (P)), then for any $p \in X$ one has $Y \times_X \operatorname{Spec}(\mathcal{O}_{X,p})$ satisfying (P) affine locally.*

Proof. Suppose U_i is an affine open cover for X with $i \in I$. Let $f^{-1}(U_i)$ have an affine open cover given by V_j where $j \in J_i$. So to check this, without loss of generality, we can replace the schemes in question with those of the form $U_i = \operatorname{Spec}(R)$ and $V_j = \operatorname{Spec}(B)$ for $j \in J$. It follows that $V_j = \operatorname{Spec}(B \otimes_R S^{-1}R)$ (where $S \subseteq R$ is multiplicatively closed). However, $B \otimes_R S^{-1}R$ is isomorphic to $U^{-1}B$ as a B-algebra where U is some multiplicatively closed subset, so the desired claim follows. □

Theorem 0.2 (cf. [CR15, Theorem 1.1]). *Let $f\colon Y \to X$ be a locally projective birational morphism of quasi-compact regular schemes. Then $\mathcal{O}_X \xrightarrow{ntrl.} \mathbf{R}f_*\mathcal{O}_Y$ is an isomorphism.*

Proof. The unit morphism of $\mathcal{O}_X$ with respect to derived pushforward and derived pullback adjunction factors via natural morphisms

$$\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y \longrightarrow \mathbf{R}f_*\mathcal{O}_Y.$$

As X is regular, we know that $\mathbf{R}f_*\mathcal{O}_Y \in \operatorname{Perf}(X)$, so [LV25, Lemma 3.16] tells us $\mathcal{O}_X \xrightarrow{ntrl.} \mathbf{R}f_*\mathcal{O}_Y$ splits. Hence, taking 0-th cohomology sheaves, we see that $\mathcal{O}_X \xrightarrow{ntrl.} f_*\mathcal{O}_Y$ splits. Then $\mathcal{O}_X \xrightarrow{ntrl.} f_*\mathcal{O}_Y$ is an isomorphism. So, the second claim follows if we can check $f_*\mathcal{O}_Y \xrightarrow{ntrl.} \mathbf{R}f_*\mathcal{O}_Y$ is an isomorphism. However, this is equivalent to showing the first claim (e.g. $\mathbf{R}f_*\mathcal{O}_Y$ is concentrated in degree zero).

Now we show the first claim. Since f is locally projective, there is an affine open cover of X such that the base change of f along the open immersion of each component is projective. In particular, each such base change of f is a projective birational morphism of quasi-compact regular schemes. Note that $\mathbf{R}f_*\mathcal{O}_Y$ being concentrated in degree zero is

Date: December 18, 2025.

equivalent to the restriction of $\mathbf{R}f_*\mathcal{O}_Y$ to each component satisfying the same condition. So, we can impose X be affine and f be projective.

Observe for each $p \in X$ the base change of f along $\operatorname{Spec}(\mathcal{O}_{X,p}) \xrightarrow{ntrl.} X$ is a projective birational morphism of quasi-compact regular schemes. Indeed, the base change of f along $\operatorname{Spec}(\mathcal{O}_{X,p}) \xrightarrow{ntrl.} X$ is projective, and the stalks of X are regular local rings. Furthermore, Lemma 0.1 tells us that $Y \times_X \operatorname{Spec}(\mathcal{O}_{X,p})$ is a regular scheme. Recall that generalizations lift along flat morphisms of schemes (see e.g. [Sta25, Tag 03HV]). Hence, [GD71, §1, Proposition 3.9.9] implies the base change of f along $\operatorname{Spec}(\mathcal{O}_{X,p}) \xrightarrow{ntrl.} X$ is birational, justifying the claimed observation. Fortunately, the condition that $\mathbf{R}f_*\mathcal{O}_Y$ is concentrated at degree zero can be checked at stalks. Then the observation allows us to further impose X is the affine spectrum of a regular local ring R.

Lastly, $\mathbf{R}f_*\mathcal{O}_Y$ being concentrated at degree zero is equivalent to its (derived) pullback along $R \to \widehat{R}$ satisfying the same condition. So, if we can check this latter condition, then we complete the proof. By [Sta25, Tag 0BG6], the base change of Y along $R \to \widehat{R}$ is regular. Furthermore, [GD71, §1, Proposition 3.9.9] implies the base change of f along $R \to \widehat{R}$ is birational. Clearly, the base change of f along $R \to \widehat{R}$ is projective. Note that $\widehat{R}$ is a regular complete local domain (see e.g. [Sta25, Tag 00NP & Tag 07NY]). Tying this together, the base change of f along $R \to \widehat{R}$ is a projective birational morphism of quasi-compact regular schemes. However, $\widehat{R}$ is excellent (see e.g. [Sta25, Tag 07QW]), and so the latter claim follows via [CR15, Theorem 1.1]. □

Remark 0.3. This flavor of reduction in the proof of Theorem 0.2 to (complete) local rings for studying proper birational morphisms of Noetherian schemes has appeared before (see e.g. [Mur25, Theorem 8.2] or [Čes21, §1.13]).

The case of Theorem 0.2 where X is of finite type over a field of characteristic zero and f is projective was initially shown by Hironaka [Hir64b, (2), pg. 144], whereas a similar argument for the case where X is excellent of Krull dimension two and f is projective was given by Lipman [Lip69, Proposition 1.2]. Later, this was extended to the case where X is of finite type over a perfect field and f is projective by Chatzistamatiou–Rülling [CR11, Corollary 3.2.10] using Hodge-theoretic techniques. Shortly thereafter, the same authors further improved their result to the case where X is excellent, admits a dualizing complex, and f is projective [CR15, Theorem 1.1 & Conventions], employing arguments based on Grothendieck duality.

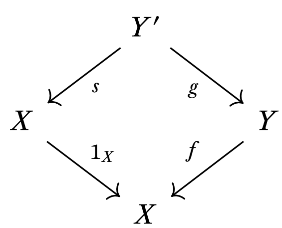
Corollary 0.4. *Let $f\colon Y \to X$ be a proper birational morphism of quasi-compact regular schemes. Then $\mathbf{R}^j f_*\mathcal{O}_Y = 0$ for $j > 0$ iff f factors through a locally projective birational morphism from a regular scheme to X.*

Proof. Our hypothesis ensures there is a locally projective birational morphism $g\colon X' \to X$ from a regular scheme and a morphism $h\colon X' \to X$ such that $h = f \circ g$. By arguing affine locally with [Sta25, Tag 0C4Q], we know that g is locally projective. Moreover, as h and f are birational, we see that g must be as well. Hence, g is a locally projective proper birational morphism between quasi-compact regular schemes. Then Theorem 0.2 implies $\mathcal{O}_Y \xrightarrow{ntrl.} \mathbf{R}g_*\mathcal{O}_{X'}$ and $\mathcal{O}_X \xrightarrow{ntrl.} \mathbf{R}h_*\mathcal{O}_{X'}$ are isomorphisms in their respective derived categories. However, $h = f \circ g$, and so $\mathcal{O}_X \xrightarrow{ntrl.} \mathbf{R}f_*\mathcal{O}_Y$ is an isomorphism, which completes the proof. □

The hypothesis of Corollary 0.4 is expected to be very mild in practice. Specifically, [Gro65, Remarque (7.9.6)] conjectures that every quasi-compact, quasi-excellent, reduced scheme admits a projective birational morphism from a regular scheme. This existence is known in some cases, e.g. quasi-excellent $\mathbb{Q}$-schemes [Hir64b, Hir64a, Tem08] and quasi-compact excellent schemes of Krull dimension two [Lip78]. It is also worth mentioning [Česv21], which provides a variation of [Gro65, Remarque (7.9.6)] showing that certain classes of schemes admit projective birational morphisms from a Cohen–Macaulay scheme.

Proposition 0.5 (cf. [CR15, Question (i), pg. 2133]). *Assume* [Gro65, Remarque (7.9.6)] *is true. Then for every proper birational morphism $f: Y \to X$ of quasi-compact regular schemes one has $\mathbf{R}^j f_*\mathcal{O}_Y = 0$ if $j > 0$.*

Proof. A similar argument to Theorem 0.2 allows one to reduce to the case where X is the affine spectrum of a Noetherian complete local ring. From [Lüt93, Lemma 2.2], there is a commutative diagram



where s is an admissible blowup of X along the largest open subscheme of X that f is an isomorphism over. Now [Gro65, Remarque (7.9.6)] implies there is a projective birational morphism $h: \widetilde{Y} \to Y'$ from a regular scheme. Observe that the composition $s \circ h = f \circ g \circ h$ is projective, and so, $g \circ h$ must be projective (see e.g. [Sta25, Tag 0C4Q]). Clearly, $g \circ h$ is a birational morphism, so the desired claim follows from Corollary 0.4. □

Corollary 0.6 (cf. [CR15, Question (ii), pg. 2133]). *Assume* [Gro65, Remarque (7.9.6)] *is true. Let S be a Noetherian scheme. Suppose $f_i: Y_i \to S$ are quasi-compact quasi-excellent regular S-schemes. If X and Y are properly S-birational, then $\mathbf{R}(f_1)_*\mathcal{O}_{Y_1} \cong \mathbf{R}(f_2)_*\mathcal{O}_{Y_2}$.*

Proof. Our hypothesis gives us a proper birational morphisms $h_i: Y \to Y_i$ such that $f_1 \circ h_1 = f_2 \circ h_2$. Now [Gro65, Remarque (7.9.6)] ensures there is a projective resolution $g: \widetilde{Y} \to Y$ from a regular scheme. Hence, each $h_i \circ g$ is a proper birational morphism of quasi-compact regular schemes. Then Proposition 0.5 tells us $\mathcal{O}_{Y_i} \xrightarrow{ntrl.} \mathbf{R}(h_i \circ g)_*\mathcal{O}_{\widetilde{Y}}$ is an isomorphism for each i, and so

$$\mathbf{R}(f_1)_*\mathcal{O}_{Y_1} \cong \mathbf{R}(f_1 \circ h_1 \circ g)_*\mathcal{O}_{\widetilde{Y}} \cong \mathbf{R}(f_2 \circ h_2 \circ g)_*\mathcal{O}_{\widetilde{Y}} \cong \mathbf{R}(f_2)_*\mathcal{O}_{Y_2}.$$

This completes the proof. □

Remark 0.7.

(1) Observe that the proof of Proposition 0.5 and Corollary 0.6 do not require [Gro65, Remarque (7.9.6)] to hold in full generality if the schemes in question have finite Krull dimension. Instead, it suffices for the claim to be true for quasi-compact quasi-excellent reduced schemes of suitable Krull dimension.

(2) [CR15, Question (iii), pg. 2133] asks whether one can impose singularities on X and still obtain results similar to those discussed here. Such singularities that may be approachable via methods of this note should exhibit suitable local behavior (e.g. stability under completion). Variants of Question (iii) seeking to relax the singularity assumptions on Y are less clear, see [IiY24, Example 1.6].

References

[Čes21] Kęstutis Česnavičius. Macaulayfication of Noetherian schemes. *Duke Math. J.*, 170(7):1419–1455, 2021.

[CR11] Andre Chatzistamatiou and Kay Rülling. Higher direct images of the structure sheaf in positive characteristic. *Algebra Number Theory*, 5(6):693–775, 2011.

[CR15] Andre Chatzistamatiou and Kay Rülling. Vanishing of the higher direct images of the structure sheaf. *Compos. Math.*, 151(11):2131–2144, 2015.

[GD71] Alexander Grothendieck and Jean A. Dieudonné. *Éléments de géométrie algébrique. I*, volume 166 of *Grundlehren Math. Wiss.* Springer, Cham, 1971.

[Gro65] Alexander Grothendieck. éléments de géométrie algébrique : IV. étude locale des schémas et des morphismes de schémas, seconde partie. *Publications Mathématiques de l'IHÉS*, 24:5–231, 1965.

[Hir64a] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I. *Ann. Math. (2)*, 79:109–203, 1964.

[Hir64b] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. II. *Ann. Math. (2)*, 79:205–326, 1964.

[IiY24] Shihoko Ishii and Ken ichi Yoshida. On vanishing of higher direct images of the structure sheaf. arXiv:2410.15282, 2024. to appear in Michigan Mathematics Journal.

[Lip69] J. Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Publ. Math., Inst. Hautes Étud. Sci.*, 36:195–279, 1969.

[Lip78] Joseph Lipman. Desingularization of two-dimensional schemes. *Ann. Math. (2)*, 107:151–207, 1978.

[Lüt93] W. Lütkebohmert. On compactification of schemes. *Manuscr. Math.*, 80(1):95–111, 1993.

[LV25] Pat Lank and Sridhar Venkatesh. Triangulated characterizations of singularities. *Nagoya Mathematical Journal*, page 1–15, 2025.

[Mur25] Takumi Murayama. Relative vanishing theorems for $\mathbb{Q}$-schemes. *Algebr. Geom.*, 12(1):84–144, 2025.

[Sta25] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2025.

[Tem08] Michael Temkin. Desingularization of quasi-excellent schemes in characteristic zero. *Adv. Math.*, 219(2):488–522, 2008.

P. Lank, Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, Via Cesare Saldini 50, 20133 Milano, Italy

Email address: plankmathematics@gmail.com