GORENSTEINNESS IS FOURIER–MUKAI INVARIANT: A CLASSICAL APPROACH

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We prove the following. Its proof closely follows the scheme argument for [Sta26].

Proposition 0.1. *Let $f_i\colon Y_i \to S$ be proper flat morphisms of Noetherian algebraic spaces. Suppose Y_1 and Y_2 are Fourier–Mukai S-partners given by a kernel $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Then f_1 is Gorenstein if, and only if, f_2 is Gorenstein.*

Lemma 0.2. *Let $f\colon X \to \mathrm{Spec}(k)$ be a proper morphism from an algebraic space to a field. Then $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is a dualizing complex. Moreover, for $E \in D_{\mathrm{qc}}(X)$ there are functorial isomorphisms which are compatible with shifts and distinguished triangles,*

$$\mathrm{Ext}^i(E, f^!\mathcal{O}_{\mathrm{Spec}(k)}) \cong \mathrm{Hom}(H^{-i}(\mathcal{X}, E), k)$$

Proof. By [GL26b, Lemma 3.3], $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is a dualizing complex on X.

The last part holds by construction as $f^\times$ is the right adjoint to $\mathbf{R}f_*\colon D_{\mathrm{qc}}(X) \to D(\mathcal{O}_{\mathrm{Spec}(k)}) = D(k)$ which we can identify with $K \mapsto \mathbf{R}\Gamma(X_{\text{étale}}, K)$. We also use that the derived category $D(k)$ of k-modules is the same as the category of graded k-vector spaces. In particular, properness of f yields a functorial isomorphism $f^\times \to f^!$ on D_{qc}, see [GL26a, Lemma 2.22]. There exists a string of canonical isomorphisms

$$\begin{aligned}
\mathrm{Ext}^i(E, f^!\mathcal{O}_{\mathrm{Spec}(k)}) &\cong \mathrm{Hom}(E, f^!\mathcal{O}_{\mathrm{Spec}(k)}[i]) \\
&\cong \mathrm{Hom}(E, f^\times\mathcal{O}_{\mathrm{Spec}(k)}[i]) \\
&\cong \mathrm{Hom}(\mathbf{R}f_*E, \mathcal{O}_{\mathrm{Spec}(k)}[i]) \\
&\cong \mathrm{Hom}(\mathbf{R}\Gamma(X_{\text{étale}}, E), k[i]) \\
&\cong \mathrm{Hom}(H^{-i}(X, E), k).
\end{aligned}$$

The last isomorphism occurs because k-linear duality is exact. This gives functoriality, and compatibility with cones and shifts. □

Lemma 0.3. *Let $f\colon X \to \mathrm{Spec}(k)$ be a proper morphism from an algebraic space to a field. Choose $E \in D_{\mathrm{qc}}(X)$ and $P \in \mathrm{Perf}(X)$. There exist functorial isomorphisms, compatible with shifts and distinguished triangles, for all $i \in \mathbb{Z}$:*

$$\mathrm{Ext}^i(E, f^!\mathcal{O}_{\mathrm{Spec}(k)} \otimes^{\mathbf{L}} P) \cong \mathrm{Ext}^{-i}(P, E)^\vee.$$

If $E \in D^b_{\mathrm{coh}}(X)$, then this is an isomorphism of finite dimensional k-vector spaces.

Proof. Since P is perfect, there exists an isomorphism for all $A, B \in D(X)$,

$$\mathbf{R}\,\mathrm{Hom}(A \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(P, \mathcal{O}_X), B) \to \mathbf{R}\,\mathrm{Hom}(A, P \otimes^{\mathbf{L}} B).$$

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This uses the identification (see [Sta26, Tag 0B6E])

$$\mathbf{R}\Gamma(X_{\text{étale}}, \mathbb{R}\mathcal{H}om(-,-)) = \mathbf{R}\operatorname{Hom}(-,-),$$

and applying $\mathbf{R}\Gamma(X_{\text{étale}}, -)$ to the functorial isomorphism

$$\begin{aligned}
&\mathbb{R}\mathcal{H}om(A \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(P, \mathcal{O}_X), B) \\
&\quad \cong \mathbb{R}\mathcal{H}om(A, \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(P, \mathcal{O}_X), B)) && \text{[Sta26, Tag 08J9]} \\
&\quad \cong \mathbb{R}\mathcal{H}om(A, \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(P, \mathcal{O}_X), \mathcal{O}_X) \otimes^{\mathbf{L}} B) && \text{[Sta26, Tag 08JJ]} \\
&\quad \cong \mathbb{R}\mathcal{H}om(A, P \otimes^{\mathbf{L}} B).
\end{aligned}$$

Thus, for all $i \in \mathbb{Z}$, we obtain

$$\begin{aligned}
\operatorname{Ext}^i(E, f^!\mathcal{O}_{\operatorname{Spec}(k)} \otimes^{\mathbf{L}} P) &\cong \operatorname{Ext}^i(E \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(P, \mathcal{O}_X), f^!\mathcal{O}_{\operatorname{Spec}(k)}) \\
&\cong H^{-i}(X, E \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(P, \mathcal{O}_X))^\vee && \text{(Lemma 0.2)} \\
&\cong H^{-i}(X, \mathbb{R}\mathcal{H}om(P, E))^\vee && \text{[Sta26, Tag 08JJ]} \\
&\cong (H^{-i}\mathbf{R}\operatorname{Hom}(P, E))^\vee \\
&\cong \operatorname{Ext}^{-i}(P, E)^\vee.
\end{aligned}$$

The last claim follows from [Sta26, Tag 0DoT]. □

Remark 0.4. Let $f : X \to \operatorname{Spec}(k)$ be a proper morphism from an algebraic space to a field. By Lemma 0.2, the identity of $f^!\mathcal{O}_{\operatorname{Spec}(k)}$ corresponds to a morphism

$$t : H^0(X, f^!\mathcal{O}_{\operatorname{Spec}(k)}) \to k.$$

Let $K \in D_{\text{qc}}(X)$. Observe that the identification $\operatorname{Hom}(K, f^!\mathcal{O}_{\operatorname{Spec}(k)})$ with $H^0(X, K)^\vee$ in Lemma 0.2 corresponds to the pairing

$$\operatorname{Hom}(K, f^!\mathcal{O}_{\operatorname{Spec}(k)}) \times H^0(X, K) \to k, (\alpha, \beta) \mapsto t(\alpha(\beta)).$$

This follows from the functoriality of the isomorphisms. Similarly, for any $i \in \mathbf{Z}$, we get the pairing

$$\operatorname{Ext}^i(K, f^!\mathcal{O}_{\operatorname{Spec}(k)}) \times H^{-i}(X, K) \to k, (\alpha, \beta) \mapsto t(\alpha(\beta)).$$

Here, α is treated as a morphism $K[-i] \to f^!\mathcal{O}_{\operatorname{Spec}(k)}$, and β as an element of $H^0(X, K[-i])$ in order to define $\alpha(\beta)$. For arbitrary $K \in D_{\text{qc}}(X)$, the first adjoint of this pairing gives an isomorphism

$$\operatorname{Ext}^i(K, f^!\mathcal{O}_{\operatorname{Spec}(k)}) \to (H^{-i}(X, K))^\vee.$$

Consequently, the pairing is nondegenerate in both variables. If $K \in D^b_{\text{coh}}(\mathcal{O}_X)$, then $\operatorname{Ext}^i(K, f^!\mathcal{O}_{\operatorname{Spec}(k)})$ and $H^i(X, K)$ are finite dimensional k-vector spaces (see [Sta26, Tag 0DoS and 0DoR]), and the pairings are perfect.

Remark 0.5. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a ringed site. Choose $K, M \in D(\mathcal{O}_\mathcal{X})$. Set $A = \Gamma(X, \mathcal{O}_\mathcal{X})$. The (global) cup product in this setting is a morphism

$$\mu : \mathbf{R}\Gamma(\mathcal{X}, K) \otimes^{\mathbf{L}} \mathbf{R}\Gamma(\mathcal{X}, M) \to \mathbf{R}\Gamma(\mathcal{X}, K \otimes^{\mathbf{L}} M)$$

in $D(A)$. We define it as the relative cup product for the morphism of ringed topoi $f : (\operatorname{Sh}(\mathcal{X}), \mathcal{O}_\mathcal{X}) \to (\operatorname{Sh}(pt), A)$ as in [Sta26, Tag 0B6C] via $D(pt, A) = D(A)$. This morphism defines pairings

$$\cup : H^i(\mathcal{X}, K) \times H^j(\mathcal{X}, M) \to H^{i+j}(\mathcal{X}, K \otimes^{\mathbf{L}} M).$$

In particular, let $\xi \in H^i(\mathcal{X}, K) = H^i(\mathbf{R}\Gamma(\mathcal{X}, K))$ and $\eta \in H^j(\mathcal{X}, M) = H^j(\mathbf{R}\Gamma(\mathcal{X}, M))$. It is possible to 'tensor' to get an element $\xi \otimes \eta$ in $H^{i+j}(\mathbf{R}\Gamma(\mathcal{X}, K) \otimes^{\mathbf{L}} \mathbf{R}\Gamma(\mathcal{X}, M))$ [Sta26, Tag 068G]. Then we apply μ to get the desired element $\xi \cup \eta = \mu(\xi \otimes \eta)$ of $H^{i+j}(\mathcal{X}, K \otimes^{\mathbf{L}} M)$.

Here is another way to think of the cup product of ξ and η. Namely, we can write

$$H^i(\mathbf{R}\Gamma(\mathcal{X}, K)) = \mathrm{Hom}(\mathcal{O}_{\mathcal{X}}[-i], K) \quad \text{and} \quad H^j(\mathbf{R}\Gamma(\mathcal{X}, M)) = \mathrm{Hom}(\mathcal{O}_{\mathcal{X}}[-j], M)$$

because $\mathrm{Hom}(\mathcal{O}_{\mathcal{X}}, -) = \Gamma(\mathcal{X}, -)$. Thus ξ and η are the 'same' thing as morphisms

$$\tilde{\xi} \colon \mathcal{O}_{\mathcal{X}}[-i] \to K \quad \text{and} \quad \tilde{\eta} \colon \mathcal{O}_{\mathcal{X}}[-j] \to M$$

Combining this with the functoriality of the derived tensor product we obtain

$$\mathcal{O}_{\mathcal{X}}[-i-j] = \mathcal{O}_{\mathcal{X}}[-i] \otimes^{\mathbf{L}} \mathcal{O}_{\mathcal{X}}[-j] \xrightarrow{\tilde{\xi} \otimes \tilde{\eta}} K \otimes^{\mathbf{L}} M$$

which by the same token as above is an element of $H^{i+j}(\mathcal{X}, K \otimes^{\mathbf{L}} M)$.

Lemma 0.6. *The construction in Remark 0.5 gives the cup product.*

Proof. With $f \colon (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (pt, A)$ as above we have $\mathbf{R}f_*(-) = \mathbf{R}\Gamma(\mathcal{X}, -)$ and our morphism μ is adjoint to the morphism

$$\mathbf{L}f^*(\mathbf{R}f_*K \otimes^{\mathbf{L}} \mathbf{R}f_*M) = \mathbf{L}f^*\mathbf{R}f_*K \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}f_*M \xrightarrow{\epsilon_K \otimes \epsilon_M} K \otimes^{\mathbf{L}} M$$

where ϵ is the counit of the adjunction between $\mathbf{L}f^*$ and $\mathbf{R}f_*$. If we think of ξ and η as morphisms $\xi \colon A[-i] \to \mathbf{R}\Gamma(\mathcal{X}, K)$ and $\eta \colon A[-j] \to \mathbf{R}\Gamma(\mathcal{X}, M)$, then the tensor $\xi \otimes \eta$ corresponds to the morphism

$$A[-i-j] = A[-i] \otimes^{\mathbf{L}} A[-j] \xrightarrow{\xi \otimes \eta} \mathbf{R}\Gamma(\mathcal{X}, K) \otimes^{\mathbf{L}} \mathbf{R}\Gamma(\mathcal{X}, M).$$

By definition the cup product $\xi \cup \eta$ is the morphism $A[-i-j] \to \mathbf{R}\Gamma(\mathcal{X}, K \otimes^{\mathbf{L}} M)$ which is adjoint to

$$(\epsilon_K \otimes \epsilon_M) \circ \mathbf{L}f^*(\xi \otimes \eta) = (\epsilon_K \circ \mathbf{L}f^*\xi) \otimes (\epsilon_M \circ \mathbf{L}f^*\eta).$$

However, it is easy to see that $\epsilon_K \circ \mathbf{L}f^*\xi = \tilde{\xi}$ and $\epsilon_M \circ \mathbf{L}f^*\eta = \tilde{\eta}$. We conclude that $\widetilde{\xi \cup \eta} = \tilde{\xi} \otimes \tilde{\eta}$ which means we have the desired agreement. □

Lemma 0.7. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed site. Choose $K, E \in D(\mathcal{O}_{\mathcal{X}})$ with E perfect. The diagram*

$$\begin{array}{ccc} H^0(\mathcal{X}, K \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, \mathcal{O}_{\mathcal{X}})) \times H^0(\mathcal{X}, E) & \longrightarrow & H^0(\mathcal{X}, K \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(E, \mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} E) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(E, K) \times H^0(\mathcal{X}, E) & \longrightarrow & H^0(\mathcal{X}, K) \end{array}$$

commutes where the top horizontal arrow is the cup product, the right vertical arrow uses the morphism $\epsilon \colon \mathbb{R}\mathcal{H}om(E, \mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} E \to \mathcal{O}_{\mathcal{X}}$ [Sta26, Tag 0FPU], the left vertical arrow uses [Sta26, Tag 08JJ], and the bottom horizontal arrow is the obvious one.

Proof. We abbreviate $\otimes = \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{X}}}$ and $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$. We identify E and K with $\mathbb{R}\mathcal{H}om(\mathcal{O}, E)$ and $\mathbb{R}\mathcal{H}om(\mathcal{O}, K)$ and moreover, we identify $E^\vee$ with $\mathbb{R}\mathcal{H}om(E, \mathcal{O})$.

Let $\xi \in H^0(\mathcal{X}, K \otimes E^\vee)$ and $\eta \in H^0(\mathcal{X}, E)$. Denote $\tilde{\xi} \colon \mathcal{O} \to K \otimes E^\vee$ and $\tilde{\eta} \colon \mathcal{O} \to E$ the corresponding morphisms in $D(\mathcal{O})$. By Lemma 0.6, the cup product $\xi \cup \eta$ corresponds to $\tilde{\xi} \otimes \tilde{\eta} \colon \mathcal{O} \to K \otimes E^\vee \otimes E$.

We claim the morphism $\xi' : E \to K$ corresponding to ξ by [Sta26, Tag 08JJ] is the composition

$$E = \mathcal{O} \otimes E \xrightarrow{\tilde{\xi} \otimes 1_E} K \otimes E^\vee \otimes E \xrightarrow{1_K \otimes \epsilon} K.$$

The construction in [Sta26, Tag 08JJ] uses the evaluation morphism [Sta26, Tag 08JE] which in turn is constructed using the identification of E with $\mathbf{R}\mathcal{H}om(\mathcal{O}, E)$ and the composition $\underline{\circ}$ constructed in [Sta26, Tag 0A98]. Hence, ξ' is the composition

$$\begin{aligned} E = \mathcal{O} \otimes \mathbb{R}\mathcal{H}om(\mathcal{O}, E) &\xrightarrow{\tilde{\xi} \otimes 1} \mathbb{R}\mathcal{H}om(\mathcal{O}, K) \otimes \mathbb{R}\mathcal{H}om(E, \mathcal{O}) \otimes \mathbb{R}\mathcal{H}om(\mathcal{O}, E) \\ &\xrightarrow{\underline{\circ} \otimes 1} \mathbb{R}\mathcal{H}om(E, K) \otimes \mathbb{R}\mathcal{H}om(\mathcal{O}, E) \\ &\xrightarrow{\underline{\circ}} \mathbb{R}\mathcal{H}om(\mathcal{O}, K) = K. \end{aligned}$$

The claim follows immediately from this, the fact that the composition $\underline{\circ}$ constructed in [Sta26, Tag 0A98] is associative, and the fact that ϵ is defined as the composition $\underline{\circ} : E^\vee \otimes E \to \mathcal{O}$ in [Sta26, Tag 0FPU].

Using the results from the previous two paragraphs, we find the statement of the lemma is that $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes \tilde{\eta})$ is equal to $(1_K \otimes \epsilon) \circ (\tilde{\xi} \otimes 1_E) \circ (1_{\mathcal{O}} \otimes \tilde{\eta})$ which is immediate. □

Lemma 0.8. *Let $f : X \to \mathrm{Spec}(k)$ be a proper morphism from an algebraic space to a field. Let $t : H^0(X, f^!\mathcal{O}_{\mathrm{Spec}(k)}) \to k$ be as in Remark 0.4. Let $E \in \mathrm{Perf}(X)$. Then the pairings*

$$H^i(X, f^!\mathcal{O}_{\mathrm{Spec}(k)} \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)) \times H^{-i}(X, E) \to k, \quad (\xi, \eta) \mapsto t((1_{f^!\mathcal{O}_{\mathrm{Spec}(k)}} \otimes \epsilon)(\xi \cup \eta))$$

are perfect for all i. Here $\cup$ denotes the cup product of Remark 0.5 and $\epsilon : \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X) \otimes^{\mathbf{L}} E \to \mathcal{O}_X$ is as in [Sta26, Tag 0FPU].

Proof. We can reduce to the case $i = 0$ after replacing E with $E[-i]$. By Lemma 0.7, the pairing is the same as the one discussed in Remark 0.4. Hence, the result follows. □

Definition 0.9. Let k be a field. Let $\mathcal{T}$ be a k-linear triangulated category such that $\dim_k \mathrm{Hom}(X, Y) < \infty$ for all $X, Y \in \mathcal{T}$. A k-linear autoequivalence $S : \mathcal{T} \to \mathcal{T}$ is called a **Serre functor** if there exist k-linear isomorphisms

$$c_{X,Y} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, S(X))^\vee$$

which are functorial in $X, Y \in \mathcal{T}$.

Lemma 0.10. *Let X be a Gorenstein algebraic space. Then any dualizing complex K on X is an invertible object in $D(X)$ (see* [Sta26, Tag 0FFN]*). In particular, K is a perfect complex.*

Proof. Let K be a dualizing complex on X. Choose an étale presentation $s : U \to X$ by an affine scheme. Then $\mathbf{L}s^*K$ is a dualizing complex. Since U is Gorenstein, [Sta26, Tag 0BFQ] says $\mathbf{L}s^*K$ is invertible. Thus, by [GL26b, Lemma 3.1], K is invertible. □

Lemma 0.11. *Let k be a field. Consider a proper morphism $f : X \to \mathrm{Spec}(k)$ of algebraic spaces. If X is Gorenstein, then the functor*

$$S : \mathrm{Perf}(X) \to \mathrm{Perf}(X), P \mapsto (f^!\mathcal{O}_{\mathrm{Spec}(k)} \otimes^{\mathbf{L}} P)$$

is a Serre functor.

Proof. By [Sta26, Tag 0DoT], $\dim_k \mathrm{Hom}(P,Q) < \infty$ for all $P,Q \in \mathrm{Perf}(X)$. Moreover, from [GL26b, Lemma 3.3], $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is a dualizing complex. Since X is Gorenstein, Lemma 0.10 says $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is invertible. Then S is a well-defined endofunctor on $\mathrm{Perf}(X)$. Furthermore, $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ being invertible implies S is an autoequivalence. We have to find k-linear isomorphisms

$$c_{P,Q}\colon \mathrm{Hom}(P,Q) \to \mathrm{Hom}(Q,S(P))^\vee$$

which are functorial in $P,Q \in \mathrm{Perf}(X)$. To do so, we use the functorial isomorphisms

$$\mathrm{Hom}(P,Q) \cong H^0(X, Q \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(P,\mathcal{O}_X))$$

and

$$\mathrm{Hom}(Q, P \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)}) \cong H^0(X, P \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)} \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(Q,\mathcal{O}_X))$$

where $\mathbb{R}\mathcal{H}om$ is the internal Hom in $D(X)$. See [Sta26, Tag 08JJ]. As P is perfect, [Sta26, Tag 08JJ and oDoS] imply

$$\mathbb{R}\mathcal{H}om(P,\mathcal{O}_X) \cong \mathbf{R}\mathcal{H}om(P,\mathcal{O}_X)$$

where $\mathbf{R}\mathcal{H}om$ is the internal Hom for $D_{\mathrm{qc}}(X)$, and similarly for Q. Note that there exists functorial isomorphisms

$$\mathbf{R}\mathcal{H}om\left((Q \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(P,\mathcal{O}_X)),\mathcal{O}_X\right) \cong \mathbf{R}\mathcal{H}om\left(\mathbf{R}\mathcal{H}om(P,\mathcal{O}_X),\mathcal{O}_X\right) \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(Q,\mathcal{O}_X).$$

By the functorial isomorphism

$$P \to \mathbf{R}\mathcal{H}om\left(\mathbf{R}\mathcal{H}om(P,\mathcal{O}_X),\mathcal{O}_X\right),$$

the k-vector spaces above are naturally dual. See Lemma 0.8. This yields the desired isomorphisms $c_{P,Q}$, which completes the proof. □

Lemma 0.12. *Let k be a field. Consider a proper morphism $f\colon X \to \mathrm{Spec}(k)$ of algebraic spaces. If there exists a Serre functor $S\colon \mathrm{Perf}(X) \to \mathrm{Perf}(X)$, then X is Gorenstein.*

Proof. Let $P,Q \in \mathrm{Perf}(X)$. By Lemma 0.3 and the Serre functor S, there exists a functorial isomorphism for all $n \in \mathbb{Z}$,

$$\mathrm{Hom}(Q,S(P)[n]) \cong \mathrm{Hom}(P[n],Q)^\vee \cong \mathrm{Hom}(Q, P \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)}[n]).$$

The identity on $S(P)$ corresponds uniquely to a morphism $\phi\colon S(P) \to P \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)}$. Pick a compact generator G for $D_{\mathrm{qc}}(X)$. Apply $\mathrm{Hom}(G,-)$ to the distinguished triangle

$$S(P) \xrightarrow{\phi} P \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)} \to \mathrm{cone}(\phi) \to S(P)[1].$$

There exists a long exact sequence (see [Sta26, Tag 0149]),

$$\cdots \to \mathrm{Hom}(G,S(P)) \xrightarrow{\mathrm{Hom}(G,\phi)} \mathrm{Hom}(G, P \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)}) \to \mathrm{Hom}(G,\mathrm{cone}(\phi)) \to \cdots.$$

By the functorial isomorphism above, it follows that $\mathrm{Hom}(G,\mathrm{cone}(\phi)[n]) \cong 0$ for all $n \in \mathbb{Z}$. Since G compactly generates $D_{\mathrm{qc}}(X)$, we obtain that $\mathrm{cone}(\phi) \cong 0$. In other words, ϕ is an isomorphism. Taking $P := \mathcal{O}_X$ shows that $S(\mathcal{O}_X) \cong f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is perfect. By [Sta26, Tag 0E51, 0A97, and oDoS], the natural morphism

$$\mathcal{O}_X \to \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathcal{O}_X, f^!\mathcal{O}_{\mathrm{Spec}(k)}), f^!\mathcal{O}_{\mathrm{Spec}(k)})$$

is an isomorphism. Moreover, as $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is perfect, [Sta26, Tag 08JJ] implies

$$\begin{aligned}&\mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathcal{O}_X, f^!\mathcal{O}_{\mathrm{Spec}(k)}), f^!\mathcal{O}_{\mathrm{Spec}(k)})\\ &\cong \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathcal{O}_X, f^!\mathcal{O}_{\mathrm{Spec}(k)}), \mathcal{O}_X) \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)}\\ &\cong \mathbf{R}\mathcal{H}om(f^!\mathcal{O}_{\mathrm{Spec}(k)}, \mathcal{O}_X) \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)}.\end{aligned}$$

Hence, $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is an invertible object. Choose an étale presentation $s\colon U \to X$. Note that $\mathbf{L}s^*f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is a dualizing complex for U. Moreover, derived pullback is monoidal, and so this dualizing complex is invertible. Thus, [Sta26, Tag 0BFQ] says U is Gorenstein, and hence, X must be Gorenstein. □

Remark 0.13. There is an alternative proof in Lemma 0.12 to show $f^!\mathcal{O}_{\mathrm{Spec}(k)}$ is invertible. There exists an isomorphism

$$\begin{aligned}\mathcal{O}_X &\cong \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathcal{O}_X, f^!\mathcal{O}_{\mathrm{Spec}(k)}), f^!\mathcal{O}_{\mathrm{Spec}(k)}) && ([\text{Sta26, Tag 0E51}])\\ &\cong \mathbf{R}\mathcal{H}om(f^!\mathcal{O}_{\mathrm{Spec}(k)}, f^!\mathcal{O}_{\mathrm{Spec}(k)})\\ &\cong \mathbf{R}\mathcal{H}om(f^!\mathcal{O}_{\mathrm{Spec}(k)}, \mathcal{O}_X) \otimes^{\mathbf{L}} f^!\mathcal{O}_{\mathrm{Spec}(k)} && ([\text{Sta26, Tag 08JJ and 0DoS}]).\end{aligned}$$

Thus, the claim follows.

Lemma 0.14. *Let k be a field. Consider a proper morphism $f\colon X \to \mathrm{Spec}(k)$ of algebraic spaces. Then X is Gorenstein if, and only if, there exists a Serre functor $S\colon \mathrm{Perf}(X) \to \mathrm{Perf}(X)$.*

Proof. This follows from Lemmas 0.11 and 0.12. □

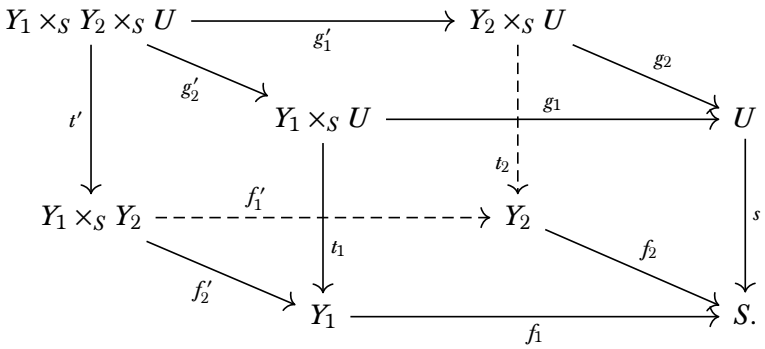
Lemma 0.15. *Let $f_i\colon Y_i \to \mathrm{Spec}(k)$ be proper morphisms of algebraic spaces over a field. Suppose $K \in D^b_{\mathrm{coh}}(Y_1 \times_k Y_2)$ is relatively perfect over each Y_i and Φ_K restricts to an equivalence $D^b_{\mathrm{coh}}(Y_1) \to D^b_{\mathrm{coh}}(Y_2)$. Then Y_1 is Gorenstein if, and only if, Y_2 is Gorenstein.*

Proof. By symmetry, it suffices to prove the case where Y_1 is Gorenstein. As Y_1 is Gorenstein, Lemma 0.14 says $\mathrm{Perf}(Y_1)$ has a Serre functor S_{Y_1}. Applying [GL26a, Lemma 10.9], Φ_K restricts to an equivalence on Perf. Hence, we can induce a Serre functor on $\mathrm{Perf}(Y_2)$; namely,

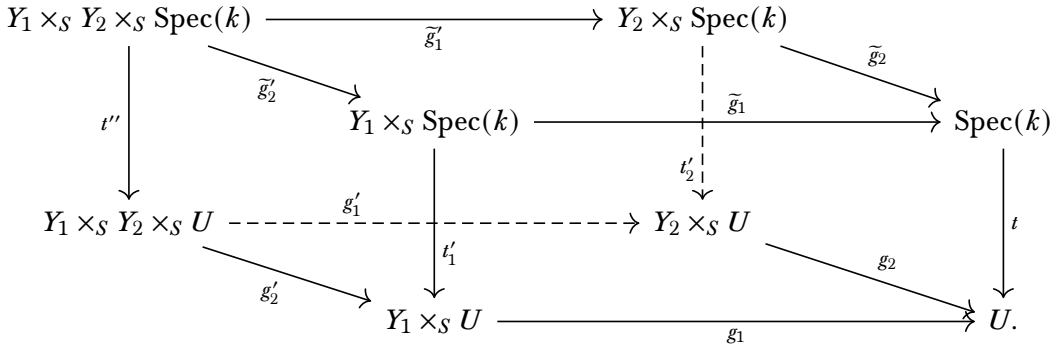
$$S_{Y_2} := \Phi_K \circ S_{Y_1} \circ (\Phi_K)^{-1}\colon \mathrm{Perf}(Y_2) \to \mathrm{Perf}(Y_2).$$

By Lemma 0.14, Y_2 is Gorenstein. □

Proof of Proposition 0.1. By [GL26a, Remark 2.7], K is relatively perfect over each Y_i. By symmetry, it suffices to prove the case where f_1 is Gorenstein. Choose an étale presentation $s\colon U \to S$. Consider the fibered cube



By [Sta26, Tag 0E15], g_1 is Gorenstein. Choose any morphism $t\colon \operatorname{Spec}(k) \to U$ from a field. By [Sta26, Tag 0E15], it suffices to check that $Y_2 \times_S \operatorname{Spec}(k)$ is Gorenstein. There exists a further fibered cube



By [GL26a, Theorems 6.16 and 10.13], $\mathbf{L}(t' \circ t'')^* K$ is relatively perfect over each $Y_i \times_S \operatorname{Spec}(k)$ and $\Phi_{\mathbf{L}(t' \circ t'')^* K}$ restricts to an equivalence on D^b_{coh}. Base change implies each $\widetilde{g}_i$ is proper. Thus, Lemma 0.15 implies $Y_2 \times_S \operatorname{Spec}(k)$ is Gorenstein, which completes the proof. □

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