FULL FAITHFUL LOCI FOR INTEGRAL TRANSFORMS ON ALGEBRAIC STACKS

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We record an openness consequence of the full faithfulness criterion of [CO26, Lemmas 1.8 to 1.10]. This first appeared in [GL26], see v4 on arXiv. Although loc. cit. does not formulate the corresponding locus, their criterion can be combined with support and base change to produce one. We include the argument in some detail. In particular, the compatibility of their comparison morphism with base change is carefully spelled out. See [HR17] for derived functors and derived categories on algebraic stacks. We follow [Sta26] for conventions on algebraic stacks.

Let $f\colon \mathcal{X} \to \mathcal{S}$ and $g\colon \mathcal{Y} \to \mathcal{S}$ be morphisms of concentrated algebraic stacks. Throughout this appendix, we fix an object $K \in D_{\mathrm{qc}}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$, and write $\Phi_K\colon D_{\mathrm{qc}}(\mathcal{X}) \to D_{\mathrm{qc}}(\mathcal{Y})$ for the associated integral transform. For a perfect object $P \in D_{\mathrm{qc}}(\mathcal{X})$ and $E \in D_{\mathrm{qc}}(\mathcal{X})$, set $\mathcal{H}_f(P,E) := \mathbf{R}f_*\,\mathbf{R}\mathcal{H}om(P,E)$. At times, this notation is used for brevity. See e.g. [Lip09, pg. 88, (3.1.8)]. Here $\mathbf{R}\mathcal{H}om$ is the internal Hom for D_{qc} and $\mathbb{R}\mathcal{H}om$ is the internal Hom for $D(-)$ (i.e. derived category of modules on the lisse-étale site of an algebraic stack). We record a basic lemma.

Lemma 0.1. *Let $\mathcal{X}$ be an algebraic space. For any $E, L \in D_{\mathrm{qc}}(\mathcal{X})$, there is a canonical isomorphism*

$$\mathbf{R}\mathcal{H}om(E,L) \to Q(\mathbb{R}\mathcal{H}om(i(E),i(L))).$$

In particular, if $\mathbb{R}\mathcal{H}om(i(E),i(L))$ has quasi-coherent cohomology, then

$$\mathbf{R}\mathcal{H}om(i(E),i(L)) \cong \mathbb{R}\mathcal{H}om(i(E),i(L)).$$

This happens if E is pseudocoherent and L is locally bounded below, see [Sta26, Tag 0A8A].

Proof. Note that there exist adjunctions

$$D_{\mathrm{qc}}(\mathcal{X}) \underset{Q}{\overset{i}{\rightleftarrows}} D(\mathcal{X}) \underset{\mathbb{R}\mathcal{H}om(i(E),-)}{\overset{((-)\otimes^{\mathbf{L}} i(E))}{\rightleftarrows}} D(\mathcal{X}).$$

By composing, we obtain

$$D_{\mathrm{qc}}(\mathcal{X}) \underset{Q(\mathbb{R}\mathcal{H}om(i(E),-))}{\overset{((-)\otimes^{\mathbf{L}} i(E))\circ i}{\rightleftarrows}} D(\mathcal{X}).$$

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Using that i is monoidal, it follows that $((-)\otimes^{\mathbf{L}} i(E))\circ i$ induces an endofunctor on $D_{\mathrm{qc}}(\mathcal{X})$, whose right adjoint is $\mathbf{R}\mathcal{H}om(E,-)$. Therefore, restriction gives an adjunction

$$D_{\mathrm{qc}}(\mathcal{X}) \underset{Q(\mathbb{R}\mathcal{H}om(i(E),-))}{\overset{((-)\otimes^{\mathbf{L}} i(E))\circ i}{\rightleftarrows}} D_{\mathrm{qc}}(\mathcal{X}).$$

Thus, the desired claim follows from uniqueness of adjoints. □

Note that P being perfect ensures that $\mathbf{R}\mathcal{H}om(P,E)\cong\mathbb{R}\mathcal{H}om(P,E)$. See [HR17, Lemma 4.3(2)]. Moreover, the same proof of Lemma 0.1 applies to algebraic stacks.

The following does not explicitly appear in the literature. We add it for convenience.

Lemma 0.2. *Let $f\colon\mathcal{Y}\to\mathcal{X}$ be a morphism of quasi-compact quasi-separated algebraic stacks. Then $\mathbf{L}f^*$ is strong monoidal (i.e. $\mathbf{L}f^*\mathcal{O}_{\mathcal{X}}\cong\mathcal{O}_{\mathcal{Y}}$ and $\mathbf{L}f^*$ commutes with the derived tensor product). Moreover, for any $P\in\mathrm{Perf}(\mathcal{X})$ and $E\in D_{\mathrm{qc}}(\mathcal{X})$, there exists a natural isomorphism*

$$\mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathbf{L}f^*E)\cong\mathbf{L}f^*\,\mathbf{R}\mathcal{H}om(P,E).$$

Proof. First, we prove that $\mathbf{L}f^*$ is monoidal. Let us recall how $\mathbf{L}f^*$ is defined. We follow [HR17, §1] and refer the reader to the relevant constructions below. Choose compatible smooth hypercoverings $p_\bullet\colon U_\bullet\to\mathcal{X}$ and $q_\bullet\colon V_\bullet\to\mathcal{Y}$ as in [HR17, §1.3]$p_\bullet\colon U_\bullet\to\mathcal{X}$ and $q_\bullet\colon V_\bullet\to\mathcal{Y}$: $p_\bullet$ is constructed from smooth surjection $p_0\colon U_0\to\mathcal{X}$ from an algebraic space and $q_\bullet$ is constructed from a smooth surjection $q_0\colon V_0\to\mathcal{Y}\times_{\mathcal{X}}U_0$ from an algebraic sapce. These give rise to semisimplicial topoi $U^+_{\bullet,\text{lis-ét}},U^+_{\bullet,\text{ét}},V^+_{\bullet,\text{lis-ét}},V^+_{\bullet,\text{ét}}$ and a commutative diagram of ringed topoi,

$$\begin{array}{ccccc}
\mathcal{Y} & \xleftarrow{q^+_{\bullet,\text{lis-ét}}} & V^+_{\bullet,\text{lis-ét}} & \xrightarrow{\mathrm{res}_{V_\bullet}} & V^+_{\bullet,\text{ét}} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{f}^+_{\bullet,\text{lis-ét}}} & & \downarrow{\scriptstyle \widetilde{f}^+_{\bullet,\text{ét}}} \\
\mathcal{X} & \xleftarrow[p^+_{\bullet,\text{lis-ét}}]{} & U^+_{\bullet,\text{lis-ét}} & \xrightarrow[\mathrm{res}_{U_\bullet}]{} & U^+_{\bullet,\text{ét}}.
\end{array}$$

The morphisms $q^+_{\bullet,\text{lis-ét}},\mathrm{res}_{V_\bullet},p^+_{\bullet,\text{lis-ét}},\mathrm{res}_{U_\bullet},\widetilde{f}^+_{\bullet,\text{ét}}$ have left exact inverse image. In particular, they correspond to morphisms of ringed topoi in the sense of [Sta26]. Moreover, the derived push/pull of derived categories of modules on the ringed topoi for $q^+_{\bullet,\text{lis-ét}},\mathrm{res}_{V_\bullet},p^+_{\bullet,\text{lis-ét}},\mathrm{res}_{U_\bullet}$ restrict and yield equivalences on the derived categories D_{qc}, e.g.

$$D_{\mathrm{qc}}(\mathcal{X}) \underset{\mathbf{R}(p^+_{\bullet,\text{lis-ét}})_*}{\overset{\mathbf{L}(p^+_{\bullet,\text{lis-ét}})^*}{\rightleftarrows}} D_{\mathrm{qc}}(U^+_{\bullet,\text{lis-ét}}) \underset{\mathbf{R}(\mathrm{res}_{U_\bullet})_*}{\overset{\mathbf{L}(\mathrm{res}_{U_\bullet})^*}{\rightleftarrows}} D_{\mathrm{qc}}(U^+_{\bullet,\text{ét}}).$$

Consequently, we define $\mathbf{L}f^*$ by the composition,

$$\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*\circ\mathbf{L}(\mathrm{res}_{V_\bullet})^*\circ\mathbf{L}(\widetilde{f}^+_{\bullet,\text{ét}})^*\circ\mathbf{R}(\mathrm{res}_{U_\bullet})_*\circ\mathbf{L}(p^+_{\bullet,\text{lis-ét}})^*.$$

In this case, the derived pullbacks above are monoidal with respect to the tensor product, see [Sta26, Tag 07A4]. By [HR17, Eq. (1.10)], $\mathbf{L}f^*\mathcal{O}_{\mathcal{X}}\cong\mathcal{O}_{\mathcal{Y}}$. It suffices to check that $\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*$ and $\mathbf{R}(p^+_{\bullet,\text{lis-ét}})_*$ are monoidal with respect to tensor products. We prove the case for $\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*$ as the other is similar. Let $A,B\in D_{\mathrm{qc}}(V^+_{\bullet,\text{lis-ét}})$. Since $\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*$ and

$\mathbf{L}(q^+_{\bullet,\text{lis-ét}})^*$ form an adjoint equivalence, the counit $\mathbf{L}(q^+_{\bullet,\text{lis-ét}})^* \circ \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_* \to 1$ and unit $1 \to \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_* \circ \mathbf{L}(q^+_{\bullet,\text{lis-ét}})^*$ are isomorphisms. Hence, we obtain

$$\begin{aligned}
&\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*(A \otimes^{\mathbf{L}} B)\\
&\quad\cong \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*\big(\mathbf{L}(q^+_{\bullet,\text{lis-ét}})^*\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*A \otimes^{\mathbf{L}} \mathbf{L}(q^+_{\bullet,\text{lis-ét}})^*\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*B\big)\\
&\quad\cong \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*\mathbf{L}(q^+_{\bullet,\text{lis-ét}})^*\big(\mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*A \otimes^{\mathbf{L}} \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*B\big)\\
&\quad\cong \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*A \otimes^{\mathbf{L}} \mathbf{R}(q^+_{\bullet,\text{lis-ét}})_*B.
\end{aligned}$$

Consequently, $\mathbf{L}f^*$ is strong monoidal.

Lastly, let $P \in \mathrm{Perf}(\mathcal{X})$ and $E \in D_{\mathrm{qc}}(\mathcal{X})$. By [HR17, Lemma 4.3(2)], we have a natural isomorphism

$$\mathbf{R}\mathcal{H}om(P,E) \cong \mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} E.$$

We prove there exist natural isomorphisms

$$\mathbf{L}f^*\,\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}}) \cong \mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathbf{L}f^*\mathcal{O}_{\mathcal{X}}) \cong \mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathcal{O}_{\mathcal{Y}}).$$

Since P is perfect, we know that P is dualizable. This means there exist morphisms $e\colon \mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} P \to \mathcal{O}_{\mathcal{X}}$ and $c\colon \mathcal{O}_{\mathcal{X}} \to P \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}})$ such that $(c \otimes^{\mathbf{L}} 1_P) \circ (1_P \otimes^{\mathbf{L}} e)$ and $(1_{\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}})} \otimes^{\mathbf{L}} c) \circ (e \otimes^{\mathbf{L}} 1_{\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}})})$ are isomorphisms. By applying $\mathbf{L}f^*$ to these diagrams, we observe that $\mathbf{L}f^*P$ is dualizable, with data given by $\mathbf{L}f^*\,\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}})$, $\mathbf{L}f^*c$ and $\mathbf{L}f^*e$. However, $\mathbf{L}f^*P$ is perfect, and so we know already it is dualizable (see [HR17, Lemma 4.3]). Consequently, by uniqueness of duals, we have

$$\mathbf{L}f^*\,\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}}) \cong \mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathcal{O}_{\mathcal{Y}}).$$

Then monoidality of $\mathbf{L}f^*$ shows a string of natural isomorphisms

$$\begin{aligned}
\mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathbf{L}f^*E) &\cong \mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathcal{O}_{\mathcal{Y}}) \otimes^{\mathbf{L}} \mathbf{L}f^*E\\
&\cong \mathbf{R}\mathcal{H}om(\mathbf{L}f^*P,\mathbf{L}f^*\mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} \mathbf{L}f^*E\\
&\cong \mathbf{L}f^*\,\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} \mathbf{L}f^*E\\
&\cong \mathbf{L}f^*(\mathbf{R}\mathcal{H}om(P,\mathcal{O}_{\mathcal{X}}) \otimes^{\mathbf{L}} E)\\
&\cong \mathbf{L}f^*\,\mathbf{R}\mathcal{H}om(P,E).
\end{aligned}$$

Thus, we are done. □

Following [CO26, §1.2 to 1.4], a perfect complex $G \in D_{\mathrm{qc}}(\mathcal{X})$ is an $\mathcal{S}$-**linear generator** if the derived pullback of G along $\mathcal{X} \times_{\mathcal{S}} U \to \mathcal{X}$ is a compact generator for every morphism $U \to \mathcal{S}$ from an affine scheme. In [CO26, §1.3], the $\mathcal{S}$-linear generator condition is related to the vanishing criterion:

$$\text{‘for any } E \in D_{\mathrm{qc}}(\mathcal{X}),\ \mathbf{R}f_*\,\mathbf{R}\mathcal{H}om(G,E) \cong 0 \text{ implies } E \cong 0\text{’}.$$

For the argument below, we record conditions under which this characterization follows directly. The point is that, when passing from the vanishing criterion to an arbitrary affine-scheme-valued base change $\mathcal{T} \to \mathcal{S}$, it is useful to know that the projection $\mathcal{X}_{\mathcal{T}} \to \mathcal{X}$ has conservative pushforward. This holds when the projection is quasi-affine; in particular, it holds when $\mathcal{S}$ has quasi-affine diagonal. We first record the relevant quasi-affineness statements.

Recall that a morphism of algebraic stacks is said to be **quasi-affine** if it is representable by schemes and quasi-affine in the sense of [Sta26, Tag 04XB].

Lemma 0.3. *Consider a commutative diagram of algebraic stacks*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{h} & \mathcal{Y} \\ & \searrow^{g} & \downarrow{\scriptstyle f} \\ & & \mathcal{X}. \end{array}$$

If both g and Δ_f are quasi-affine, then h is quasi-affine.

Proof. There exists a fibered square

$$\begin{array}{ccc} \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} & \xrightarrow{p_2} & \mathcal{Y} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow[g]{} & \mathcal{X}. \end{array}$$

By base change, p_2 is quasi-affine [Sta26, Tags 0302, 045C, 03WO, & 0423]. Moreover, the morphism $(1,h)\colon \mathcal{Z} \to \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ is the base change of the diagonal $\Delta_f\colon \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ by the morphism $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ [Sta26, Tag 003O]. Again, by base change, $(1,h)\colon \mathcal{Z} \to \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$ is quasi-affine. Since $h = p_2 \circ (1,h)$ is the composition of quasi-affine morphisms, it must be itself quasi-affine [Sta26, Tags 0301 & 045B]. □

Lemma 0.4. *Let $\mathcal{X}$ be a quasi-compact algebraic stack. Then the following are equivalent:*

(1) $\mathcal{X}$ has quasi-affine diagonal
(2) there exists a quasi-affine smooth presentation of $\mathcal{X}$ from a quasi-affine scheme
(3) every morphism to $\mathcal{X}$ from a quasi-affine scheme is quasi-affine.

Proof. First, let $\mathcal{X}$ have quasi-affine diagonal. Choose a smooth presentation $s\colon U \to \mathcal{X}$ from a quasi-affine scheme. By Lemma 0.3, s is quasi-affine.

Next, let $s\colon U \to \mathcal{X}$ be a quasi-affine smooth presentation from a quasi-affine scheme. Choose a morphism $t\colon V \to \mathcal{X}$ from a quasi-affine scheme. Consider the projection morphisms $t'\colon V \times_{\mathcal{X}} U \to U$ and $s'\colon V \times_{\mathcal{X}} U \to V$. By base change, s' is quasi-affine. In particular, $V \times_{\mathcal{X}} U$ is quasi-affine. Since t' is a morphism from a quasi-affine scheme to a quasi-separated scheme, it follows that t' must be quasi-affine [Sta26, Tag 054G]. As s is smooth and surjective, we see that t is quasi-affine.

Lastly, assume every morphism to $\mathcal{X}$ from a quasi-affine scheme is quasi-affine. There is a fibered square

$$\begin{array}{ccc} U \times_{\mathcal{X}} U & \xrightarrow{s_2} & U \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s} \\ U & \xrightarrow[s]{} & \mathcal{X}. \end{array}$$

By base change, each s_i is quasi-affine. Hence, $U \times_{\mathcal{X}} U$ is a quasi-affine scheme. Consider the commutative diagram

$$\begin{array}{ccccc}
U \times_{\mathbb{Z}} U & \longrightarrow & \mathcal{X} \times_{\mathbb{Z}} U & \longrightarrow & U \\
{\scriptstyle s_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\
U \times_{\mathbb{Z}} \mathcal{X} & \xrightarrow[s_1']{} & \mathcal{X} \times_{\mathbb{Z}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow[s]{} & \mathcal{X} & \longrightarrow & \mathrm{Spec}(\mathbb{Z}).
\end{array}$$

By base change, s_1 and s_1' are smooth and surjective. Set $s' = s_1' \circ s_1$. Appealing to [Sta26, Tag 04Z1], there exists a fibered square

$$\begin{array}{ccc}
U \times_{\mathcal{X}} U & \xrightarrow{t} & U \times_{\mathbb{Z}} U \\
{\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle s'} \\
\mathcal{X} & \xrightarrow[\Delta_{\mathcal{X}}]{} & \mathcal{X} \times_{\mathbb{Z}} \mathcal{X}.
\end{array}$$

The source and target of t are quasi-affine schemes, and so, t is quasi-affine. Since s' is smooth and surjective, [Sta26, Tags 04XD & 04XS] shows that $\Delta_{\mathcal{X}}$ is quasi-affine. □

We obtain the following.

Lemma 0.5. *Assume that $\mathcal{S}$ has quasi-affine diagonal. Let $f\colon \mathcal{X} \to \mathcal{S}$ be a concentrated morphism of algebraic stacks. The following are equivalent for any $P \in \mathrm{Perf}(\mathcal{X})$:*

(1) P is an $\mathcal{S}$-linear generator
(2) for any $E \in D_{\mathrm{qc}}(\mathcal{X})$, $\mathbf{R}f_* \mathbf{R}\mathcal{H}om(P,E) \cong 0$ implies $E \cong 0$
(3) the derived pullback of P along the natural projection $\mathcal{X} \times_{\mathcal{S}} \mathrm{Spec}(R) \to \mathcal{X}$ is a compact generator for each smooth morphism $\mathrm{Spec}(R) \to \mathcal{S}$.

Proof. By definition, (1) $\implies$ (3).

We prove that (3) $\implies$ (2). Choose a smooth surjective morphism $u\colon U = \mathrm{Spec}(R) \to \mathcal{S}$ from an affine scheme. Consider the fibered square

$$\begin{array}{ccc}
\mathcal{X}_U & \xrightarrow{a} & \mathcal{X} \\
{\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\
U & \xrightarrow[u]{} & \mathcal{S}.
\end{array}$$

Choose $E \in D_{\mathrm{qc}}(\mathcal{X})$ such that $\mathbf{R}f_* \mathbf{R}\mathcal{H}om(P,E) \cong 0$. By flat base change,

$$\mathbf{L}u^* \mathbf{R}f_* \mathbf{R}\mathcal{H}om(P,E) \cong \mathbf{R}(f_U)_* \mathbf{L}a^* \mathbf{R}\mathcal{H}om(P,E).$$

Since P is perfect, it is dualizable. Write $P^\vee := \mathbf{R}\mathcal{H}om(P, \mathcal{O}_{\mathcal{X}})$. By Lemma 0.2, there are natural isomorphisms

$$\mathbf{L}a^* \mathbf{R}\mathcal{H}om(P,E) \cong \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, \mathbf{L}a^*E).$$

Thus,

$$\mathbf{R}(f_U)_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, \mathbf{L}a^*E) \cong 0.$$

For all $n \in \mathbb{Z}$, we obtain

$$\begin{aligned}\operatorname{Hom}(\mathbf{L}a^*P, \mathbf{L}a^*E[n]) &\cong \operatorname{Hom}(\mathcal{O}_U, \mathbf{R}(f_U)_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, \mathbf{L}a^*E)[n]) \\ &\cong 0.\end{aligned}$$

By (3), $\mathbf{L}a^*P$ compactly generates $D_{\mathrm{qc}}(\mathcal{X}_U)$. Hence, $\mathbf{L}a^*E \cong 0$. Since a is smooth and surjective, derived pullback along a is conservative, and so $E \cong 0$. This proves that (3) $\implies$ (2).

We are left to check that (2) $\implies$ (1). Choose any morphism $t\colon T = \operatorname{Spec}(R) \to \mathcal{S}$ from an affine scheme. Consider the fibered square

$$\begin{array}{ccc} \mathcal{X}_T & \xrightarrow{a} & \mathcal{X} \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow[t]{} & \mathcal{S}. \end{array}$$

Since $\mathcal{S}$ has quasi-affine diagonal, Lemma 0.4 shows that $t\colon T \to \mathcal{S}$ is quasi-affine. Hence, by base change, a is quasi-affine.

Choose $E \in D_{\mathrm{qc}}(\mathcal{X}_T)$ such that $\operatorname{Hom}(\mathbf{L}a^*P, E[n]) \cong 0$ for all $n \in \mathbb{Z}$. Since T is affine, we have $\mathbf{R}(f_T)_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, E) \cong 0$. Indeed, for all $n \in \mathbb{Z}$,

$$\begin{aligned}\operatorname{Hom}(\mathcal{O}_T, \mathbf{R}(f_T)_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, E)[n]) &\cong \operatorname{Hom}(\mathbf{L}a^*P, E[n]) \\ &\cong 0.\end{aligned}$$

Since $\mathcal{O}_T$ compactly generates $D_{\mathrm{qc}}(T)$, the claim follows.

As P is perfect, there are natural isomorphisms

$$\begin{aligned}\mathbf{R}\mathcal{H}om(P, \mathbf{R}a_*E) &\cong P^\vee \otimes^{\mathbf{L}} \mathbf{R}a_*E \\ &\cong \mathbf{R}a_*(\mathbf{L}a^*P^\vee \otimes^{\mathbf{L}} E) \qquad \text{(projection formula)} \\ &\cong \mathbf{R}a_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, E).\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}\mathbf{R}f_* \mathbf{R}\mathcal{H}om(P, \mathbf{R}a_*E) &\cong \mathbf{R}f_*\mathbf{R}a_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, E) \\ &\cong \mathbf{R}t_*\mathbf{R}(f_T)_* \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, E) \\ &\cong 0.\end{aligned}$$

By (2), it follows that $\mathbf{R}a_*E \cong 0$. Since a is quasi-affine, $\mathbf{R}a_*$ is conservative [HR17, Corollary 2.8]. Hence, $E \cong 0$.

Therefore, $\mathbf{L}a^*P$ generates $D_{\mathrm{qc}}(\mathcal{X}_T)$. Since $\mathbf{L}a^*P$ is perfect and $\mathcal{X}_T$ is concentrated, it is compact. Thus, $\mathbf{L}a^*P$ is a compact generator for every morphism $T = \operatorname{Spec}(R) \to \mathcal{S}$ from an affine scheme. Hence, P is an $\mathcal{S}$-linear generator. □

Remark 0.6. The definition of an $\mathcal{S}$-linear generator is stated with respect to morphisms from affine schemes. The quasi-affine diagonal hypothesis in Lemma 0.5 is used to recover this condition from the relative Hom vanishing criterion, not to establish the base change stability built into the definition.

Whenever $\Phi_K(P)$ is perfect with $P \in \operatorname{Perf}(\mathcal{X})$, there exists a natural morphism

$$\phi_{P,E}\colon \mathcal{H}_f(P,E) \to \mathcal{H}_g(\Phi_K(P), \Phi_K(E)). \tag{0.1}$$

This arises from [Sta26, Tag 001P]. Indeed, it follows from a computation for all $A \in D_{\mathrm{qc}}(\mathcal{S})$ and $B \in D_{\mathrm{qc}}(\mathcal{X})$:

$$\begin{aligned}
\Phi_K(B) \otimes^{\mathbf{L}} \mathbf{L}g^*A &= \mathbf{R}(p_2)_*(\mathbf{L}p_1^*B \otimes^{\mathbf{L}} K) \otimes^{\mathbf{L}} \mathbf{L}g^*A \\
&\cong \mathbf{R}(p_2)_*(\mathbf{L}p_1^*B \otimes^{\mathbf{L}} K \otimes^{\mathbf{L}} \mathbf{L}p_2^*\mathbf{L}g^*A) && \text{(projection formula)} \\
&\cong \mathbf{R}(p_2)_*(\mathbf{L}p_1^*B \otimes^{\mathbf{L}} K \otimes^{\mathbf{L}} \mathbf{L}p_1^*\mathbf{L}f^*A) && \text{(pseudofunctoriality)} \\
&\cong \mathbf{R}(p_2)_*(\mathbf{L}p_1^*(B \otimes^{\mathbf{L}} \mathbf{L}f^*A) \otimes^{\mathbf{L}} K) && \text{(monoidality)} \\
&\cong \Phi_K(B \otimes^{\mathbf{L}} \mathbf{L}f^*A)
\end{aligned} \tag{0.2}$$

where $p_1 \colon \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $p_2 \colon \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the projections. Then we can construct the desired morphism from the following composition for all $A \in D_{\mathrm{qc}}(\mathcal{S})$:

$$\begin{aligned}
\mathrm{Hom}(A, \mathscr{H}_f(P,E)) &\cong \mathrm{Hom}(\mathbf{L}f^*A, \mathbf{R}\mathscr{H}om(P,E)) && \text{(push/pull)} \\
&\cong \mathrm{Hom}(P \otimes^{\mathbf{L}} \mathbf{L}f^*A, E) && \text{(tensor/Hom)} \\
&\to \mathrm{Hom}(\Phi_K(P \otimes^{\mathbf{L}} \mathbf{L}f^*A), \Phi_K(E)) \\
&\cong \mathrm{Hom}(\Phi_K(P) \otimes^{\mathbf{L}} \mathbf{L}g^*A, \Phi_K(E)) \\
&\cong \mathrm{Hom}(\mathbf{L}g^*A, \mathbf{R}\mathscr{H}om(\Phi_K(P), \Phi_K(E))) && \text{(tensor/Hom)} \\
&\cong \mathrm{Hom}(A, \mathbf{R}g_* \mathbf{R}\mathscr{H}om(\Phi_K(P), \Phi_K(E))) && \text{(push/pull).}
\end{aligned}$$

The following is straightforward.

Lemma 0.7. *Let $h \colon \mathcal{W} \to \mathcal{Z}$ be a morphism of concentrated algebraic stacks. Let $Q \in D_{\mathrm{qc}}(\mathcal{Z})$ be a compact generator. Choose $P \in \mathrm{Perf}(\mathcal{W})$ satisfying the condition that for any $E \in D_{\mathrm{qc}}(\mathcal{W})$ with $\mathbf{R}h_* \mathbf{R}\mathscr{H}om(P,E) \cong 0$ implies $E \cong 0$. Then $P \otimes^{\mathbf{L}} \mathbf{L}h^*Q$ is a compact generator.*

Proof. Choose $E \in D_{\mathrm{qc}}(\mathcal{W})$ such that $\mathrm{Hom}(P \otimes^{\mathbf{L}} \mathbf{L}h^*Q, E[n]) \cong 0$ for all $n \in \mathbb{Z}$. Then for all $n \in \mathbb{Z}$:

$$\begin{aligned}
0 &\cong \mathrm{Hom}(P \otimes^{\mathbf{L}} \mathbf{L}h^*Q, E[n]) \\
&\cong \mathrm{Hom}(\mathbf{L}h^*Q, \mathbf{R}\mathscr{H}om(P, E[n])) && \text{(tensor/hom)} \\
&\cong \mathrm{Hom}(Q, \mathbf{R}h_* \mathbf{R}\mathscr{H}om(P, E[n])) && \text{(push/pull)} \\
&\cong \mathrm{Hom}(Q, \mathbf{R}h_* \mathbf{R}\mathscr{H}om(P, E)[n]).
\end{aligned}$$

Consequently, Q being a compact generator implies $\mathbf{R}h_* \mathbf{R}\mathscr{H}om(P,E) \cong 0$. By the assumption on P, it follows that $E \cong 0$. This completes the proof. □

We can give a further explicit description of (0.1). Let

$$\mu_{A,M} \colon \Phi_K(A \otimes^{\mathbf{L}} \mathbf{L}f^*M) \xrightarrow{\cong} \Phi_K(A) \otimes^{\mathbf{L}} \mathbf{L}g^*M$$

denote the $\mathcal{S}$-linearity isomorphism. There is a natural evaluation morphism

$$e_{P,E} \colon P \otimes^{\mathbf{L}} \mathbf{L}f^* \mathscr{H}_f(P,E) \to E \tag{0.3}$$

obtained from the counit

$$\mathbf{L}f^* \mathbf{R}f_* \mathbf{R}\mathscr{H}om(P,E) \to \mathbf{R}\mathscr{H}om(P,E)$$

and tensor-Hom evaluation. Then (0.1) is the adjunct of the composite

$$\begin{aligned}\mathbf{L}g^*\mathscr{H}_f(P,E)\otimes^{\mathbf{L}}\Phi_K(P) &\cong \Phi_K(P)\otimes^{\mathbf{L}}\mathbf{L}g^*\mathscr{H}_f(P,E)\\ &\xrightarrow{\mu^{-1}_{P,\mathscr{H}_f(P,E)}} \Phi_K(P\otimes^{\mathbf{L}}\mathbf{L}f^*\mathscr{H}_f(P,E))\\ &\xrightarrow{\Phi_K(e_{P,E})} \Phi_K(E).\end{aligned}$$

Lemma 0.8. *Let $\Psi\colon \mathscr{D}\to\mathscr{D}'$ be an exact functor between triangulated categories admitting small coproducts. Assume the following:*

(1) $\mathscr{D}$ admits a compact generator G
(2) Ψ admits a right adjoint R
(3) Ψ takes compact objects to compact objects.

Then Ψ is fully faithful if, and only if, for all $i\in\mathbb{Z}$ the natural map

$$\operatorname{Ext}^i_{\mathscr{D}}(G,G)\to\operatorname{Ext}^i_{\mathscr{D}'}(\Psi(G),\Psi(G))$$

is an isomorphism.

Proof. One implication follows immediately from full faithfulness. Conversely, suppose that the displayed maps are isomorphisms. Denote by $\eta\colon 1\to R\Psi$ the unit of the adjunction. Since Ψ preserves compact objects, its right adjoint R preserves small coproducts [Nee96, Theorem 5.1]. Hence, the strictly full subcategory

$$\mathscr{T} := \{E\in\mathscr{D} : \eta_E \text{ is an isomorphism}\}$$

is localizing. Under adjunction, the map

$$\operatorname{Ext}^i_{\mathscr{D}}(G,G)\to\operatorname{Ext}^i_{\mathscr{D}'}(\Psi(G),\Psi(G))$$

identifies with the morphism induced by $\eta_G\colon G\to R\Psi(G)$. It follows that

$$\operatorname{Ext}^i_{\mathscr{D}}(G,\operatorname{cone}(\eta_G))\cong 0$$

for all $i\in\mathbb{Z}$. Since G is a compact generator, $\operatorname{cone}(\eta_G)\cong 0$. Hence, $G\in\mathscr{T}$. As $\mathscr{T}$ is localizing and contains the compact generator, $\mathscr{T}=\mathscr{D}$. Thus, η is a natural isomorphism, and so Ψ is fully faithful. □

The preceding criterion admits the following relative formulation.

Lemma 0.9. *Consider the following situation:*

(1) $D_{\mathrm{qc}}(\mathcal{S})$ admits a compact generator F;
(2) $D_{\mathrm{qc}}(\mathcal{X})$ admits an $\mathcal{S}$-linear generator $G\in\operatorname{Perf}(\mathcal{X})$;
(3) Φ_K preserves perfect complexes.

Set

$$\phi := \phi_{G,G}\colon \mathbf{R}f_*\,\mathbf{R}\mathscr{H}om(G,G)\to\mathbf{R}g_*\,\mathbf{R}\mathscr{H}om(\Phi_K(G),\Phi_K(G)). \tag{0.4}$$

Then Φ_K is fully faithful if, and only if, ϕ is an isomorphism.

Proof. Suppose first that Φ_K is fully faithful. Since F is a compact generator of $D_{\mathrm{qc}}(\mathcal{S})$, it suffices to show that applying $\operatorname{Ext}^i_{\mathcal{S}}(F,-)$ to ϕ gives an isomorphism for every $i\in\mathbb{Z}$. By the construction of (0.1), the resulting morphism is identified with

$$\operatorname{Ext}^i_{\mathcal{X}}(G\otimes^{\mathbf{L}}\mathbf{L}f^*F,G)\to\operatorname{Ext}^i_{\mathcal{Y}}(\Phi_K(G\otimes^{\mathbf{L}}\mathbf{L}f^*F),\Phi_K(G)),$$

which is an isomorphism by full faithfulness. Hence, ϕ is an isomorphism.

Conversely, suppose that ϕ is an isomorphism. Tensoring (0.4) with F, and using the projection formula and $\mathcal{S}$-linearity of Φ_K, gives an isomorphism

$$\mathbf{R}f_*\,\mathbf{R}\mathcal{H}om(G, G\otimes^{\mathbf{L}}\mathbf{L}f^*F)\to \mathbf{R}g_*\,\mathbf{R}\mathcal{H}om(\Phi_K(G),\Phi_K(G\otimes^{\mathbf{L}}\mathbf{L}f^*F)).$$

Applying $\operatorname{Ext}^i_{\mathcal{S}}(F,-)$ shows that

$$\operatorname{Ext}^i_{\mathcal{X}}(G\otimes^{\mathbf{L}}\mathbf{L}f^*F, G\otimes^{\mathbf{L}}\mathbf{L}f^*F)\to \operatorname{Ext}^i_{\mathcal{Y}}(\Phi_K(G\otimes^{\mathbf{L}}\mathbf{L}f^*F),\Phi_K(G\otimes^{\mathbf{L}}\mathbf{L}f^*F))$$

is an isomorphism for every $i\in\mathbb{Z}$.

By Lemma 0.7, $G\otimes^{\mathbf{L}}\mathbf{L}f^*F$ is a compact generator of $D_{\mathrm{qc}}(\mathcal{X})$. Moreover, Φ_K admits a right adjoint, and it takes compact objects to compact objects by hypothesis. Thus, Lemma 0.8 gives the desired full faithfulness. □

We next make explicit the base change compatibility of (0.1) needed for the argument below.

Lemma 0.10. *Consider a morphism $h\colon\mathcal{T}\to\mathcal{S}$ of concentrated algebraic stacks and the fibered diagrams*

$$\begin{array}{ccc}\mathcal{X}_{\mathcal{T}} & \xrightarrow{a} & \mathcal{X}\\ {\scriptstyle f_{\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle f}\\ \mathcal{T} & \xrightarrow[h]{} & \mathcal{S}\end{array}\qquad\qquad \begin{array}{ccc}\mathcal{Y}_{\mathcal{T}} & \xrightarrow{b} & \mathcal{Y}\\ {\scriptstyle g_{\mathcal{T}}}\downarrow & & \downarrow{\scriptstyle g}\\ \mathcal{T} & \xrightarrow[h]{} & \mathcal{S}.\end{array}$$

Assume that either h is flat or both f and g are flat. Let $K_{\mathcal{T}}$ denote the pullback of K. Set $\Phi_{\mathcal{T}}:=\Phi_{K_{\mathcal{T}}}$. By tor-independent base change, there is a natural isomorphism

$$\theta\colon \mathbf{L}b^*\Phi_K\xrightarrow{\cong}\Phi_{\mathcal{T}}\mathbf{L}a^*. \tag{0.5}$$

Let $P\in D_{\mathrm{qc}}(\mathcal{X})$ be perfect such that $\Phi_K(P)$ is perfect, and let $E\in D_{\mathrm{qc}}(\mathcal{X})$. Then there are natural base change isomorphisms

$$\gamma_{P,E}\colon \mathbf{L}h^*\mathcal{H}_f(P,E)\xrightarrow{\cong}\mathcal{H}_{f_{\mathcal{T}}}(\mathbf{L}a^*P,\mathbf{L}a^*E)$$

and

$$\delta_{P,E}\colon \mathbf{L}h^*\mathcal{H}_g(\Phi_K(P),\Phi_K(E))\xrightarrow{\cong}\mathcal{H}_{g_{\mathcal{T}}}(\Phi_{\mathcal{T}}(\mathbf{L}a^*P),\Phi_{\mathcal{T}}(\mathbf{L}a^*E)),$$

where the latter uses (0.5). Moreover, the diagram

$$\begin{array}{ccc}\mathbf{L}h^*\mathcal{H}_f(P,E) & \xrightarrow{\mathbf{L}h^*(\phi_{P,E})} & \mathbf{L}h^*\mathcal{H}_g(\Phi_K(P),\Phi_K(E))\\ {\scriptstyle\gamma_{P,E}}\downarrow\cong & & \cong\downarrow{\scriptstyle\delta_{P,E}}\\ \mathcal{H}_{f_{\mathcal{T}}}(\mathbf{L}a^*P,\mathbf{L}a^*E) & \xrightarrow[\phi_{\mathbf{L}a^*P,\mathbf{L}a^*E}]{} & \mathcal{H}_{g_{\mathcal{T}}}(\Phi_{\mathcal{T}}(\mathbf{L}a^*P),\Phi_{\mathcal{T}}(\mathbf{L}a^*E))\end{array} \tag{0.6}$$

commutes.

Proof. Since the squares in the statement are tor-independent, pullback-pushforward base change gives $\mathbf{L}h^*\mathbf{R}f_*\xrightarrow{\cong}\mathbf{R}(f_{\mathcal{T}})_*\mathbf{L}a^*$ and the analogous isomorphism for g. As P is perfect,

it is dualizable. Write $P^\vee := \mathbf{R}\mathcal{H}om(P, \mathcal{O}_\mathcal{X})$. By monoidality of derived pullback, it preserves dualizable objects and their duals. Hence, there are natural isomorphisms

$$\begin{aligned}\mathbf{L}a^*\,\mathbf{R}\mathcal{H}om(P,E) &\cong \mathbf{L}a^*(P^\vee \otimes^{\mathbf{L}} E)\\ &\cong (\mathbf{L}a^*P)^\vee \otimes^{\mathbf{L}} \mathbf{L}a^*E\\ &\cong \mathbf{R}\mathcal{H}om(\mathbf{L}a^*P, \mathbf{L}a^*E).\end{aligned}$$

See also Lemma 0.2. Under these identifications, the canonical pullback-Hom morphism is an isomorphism. This gives $\gamma_{P,E}$. The same argument with (0.5) gives $\delta_{P,E}$.

It remains to check compatibility with $\phi_{P,E}$. We spell this out because the compatibility uses the adjunction structure underlying ordinary base change. Denote the ordinary base change isomorphism by

$$\vartheta_f \colon \mathbf{L}h^*\mathbf{R}f_* \xrightarrow{\cong} \mathbf{R}(f_\mathcal{T})_*\mathbf{L}a^*.$$

Denote the corresponding counits by $\epsilon_f \colon \mathbf{L}f^*\mathbf{R}f_* \to 1$ and $\epsilon_{f_\mathcal{T}} \colon \mathbf{L}f_\mathcal{T}^*\mathbf{R}(f_\mathcal{T})_* \to 1$. The defining mate compatibility gives

$$\epsilon_{f_\mathcal{T}}\mathbf{L}a^* \circ \mathbf{L}f_\mathcal{T}^*(\vartheta_f) = \mathbf{L}a^*(\epsilon_f), \tag{0.7}$$

after the standard pseudofunctorial identifications. This is the counit form of **??**. See also **??**.

Combining (0.7) with compatibility of internal Hom with pullback for a perfect first argument gives

$$e_{\mathbf{L}a^*P,\mathbf{L}a^*E} \circ (1 \otimes^{\mathbf{L}} \mathbf{L}f_\mathcal{T}^*(\gamma_{P,E})) = \mathbf{L}a^*(e_{P,E}). \tag{0.8}$$

The Fourier–Mukai base change isomorphism (0.5) is also compatible with the linearity isomorphisms. More precisely, for $A \in D_{\mathrm{qc}}(\mathcal{X})$ and $M \in D_{\mathrm{qc}}(\mathcal{S})$,

$$(\theta_A \otimes^{\mathbf{L}} 1) \circ \mathbf{L}b^*(\mu_{A,M}) = \mu^{\mathcal{T}}_{\mathbf{L}a^*A,\mathbf{L}h^*M} \circ \theta_{A\otimes^{\mathbf{L}}\mathbf{L}f^*M}. \tag{0.9}$$

This follows by writing both μ and θ using the projection formula and ordinary pullback-pushforward base change. Naturality of the projection formula and of the ordinary base change transformation shows that the two composites agree. Equivalently, after passing to adjuncts, both composites are induced by the same pullback of the relevant counit (this should be compared with **??**).

Recall that $\phi_{P,E}$ is the adjunct of the composite obtained from μ^{-1} and $e_{P,E}$. Apply $\mathbf{L}g_\mathcal{T}^* \dashv \mathbf{R}(g_\mathcal{T})_*$ to the two composites in (0.6). By (0.8) and (0.9), their adjuncts agree. Hence, the two composites themselves agree, which proves the commutativity of (0.6). □

Remark 0.11. The mate calculation in Lemma 0.10 concerns the pullback-pushforward adjunctions $\mathbf{L}f^* \dashv \mathbf{R}f_*$ and $\mathbf{L}g^* \dashv \mathbf{R}g_*$, and their base changes. In particular, the argument does not require an identification between the pullback of a right adjoint to Φ_K and the right adjoint to $\Phi_{K_\mathcal{T}}$. This is one reason why the full faithfulness criterion admits a direct base change argument. By contrast, an argument for equivalences using the counit of Φ_K and its right adjoint requires precisely such an adjoint base change comparison.

We now record the form of the base change statement.

Lemma 0.12. *Consider the situation of Lemma 0.9. Let $h \colon \mathcal{T} \to \mathcal{S}$ be a morphism of concentrated algebraic stacks. Assume that either h is flat or both f and g are flat. Then:*

(1) if Φ_K is fully faithful and $D_{\mathrm{qc}}(\mathcal{T})$ admits a compact generator, then $\Phi_{K_\mathcal{T}}$ is fully faithful;

(2) *if h is faithfully flat and quasi-compact and $\Phi_{K_{\mathcal{T}}}$ is fully faithful, then Φ_K is fully faithful.*

Proof. We give the argument from the preceding compatibility.

We first prove (2). Suppose that h is faithfully flat and quasi-compact and that $\Phi_{K_{\mathcal{T}}}$ is fully faithful. Full faithfulness and the construction of (0.1) imply that the base changed comparison

$$\phi_{\mathcal{T}} : \mathscr{H}_{f_{\mathcal{T}}}(G_{\mathcal{T}}, G_{\mathcal{T}}) \to \mathscr{H}_{g_{\mathcal{T}}}(\Phi_{K_{\mathcal{T}}}(G_{\mathcal{T}}), \Phi_{K_{\mathcal{T}}}(G_{\mathcal{T}}))$$

is an isomorphism. By Lemma 0.10, this morphism identifies with $\mathbf{L}h^*\phi$. Since faithfully flat pullback is conservative, ϕ is an isomorphism. Then Lemma 0.9 implies that Φ_K is fully faithful.

We prove (1). If Φ_K is fully faithful, then Lemma 0.9 implies that ϕ is an isomorphism. Hence, Lemma 0.10 implies that $\phi_{\mathcal{T}}$ is an isomorphism. Set $G_{\mathcal{T}} := \mathbf{L}a^*G$. Since G is an $\mathcal{S}$-linear generator, G_T is a $\mathcal{T}$-linear generator. Choose a compact generator $F_{\mathcal{T}}$ of $D_{\mathrm{qc}}(\mathcal{T})$. By Lemma 0.7, $G_{\mathcal{T}} \otimes^{\mathbf{L}} \mathbf{L}f_{\mathcal{T}}^* F_{\mathcal{T}}$ is a compact generator of $D_{\mathrm{qc}}(\mathcal{X}_{\mathcal{T}})$. Moreover, (0.5) gives $\Phi_{\mathcal{T}}(G_{\mathcal{T}}) \cong \mathbf{L}b^*\Phi_K(G)$, which is perfect. By (0.2),

$$\Phi_{\mathcal{T}}(G_{\mathcal{T}} \otimes^{\mathbf{L}} \mathbf{L}f_{\mathcal{T}}^* F_{\mathcal{T}}) \cong \Phi_{\mathcal{T}}(G_{\mathcal{T}}) \otimes^{\mathbf{L}} \mathbf{L}g_{\mathcal{T}}^* F_{\mathcal{T}},$$

which is perfect. Consequently, $\Phi_{\mathcal{T}}$ preserves perfect complexes. Applying Lemma 0.9 over $\mathcal{T}$ to the isomorphism $\phi_{\mathcal{T}}$ shows that $\Phi_{K_{\mathcal{T}}}$ is fully faithful. □

We finish by recording the openness consequence. As mentioned earlier, it is not stated in [CO26], but we spell it out.

For a perfect complex $C \in D_{\mathrm{qc}}(\mathcal{S})$ on a Noetherian algebraic stack, set

$$\mathrm{Supp}(C) := \{p \in |\mathcal{S}| : \mathbf{L}t^*C \not\cong 0 \text{ for a representative } t\colon \mathrm{Spec}(k) \to \mathcal{S}\}.$$

This condition is independent of the representative because any two representatives admit a common field extension and pullback along a field extension is conservative. Moreover, if C is bounded pseudocoherent, then $\mathrm{Supp}(C)$ is closed. Indeed, this can be checked after a smooth presentation, where it reduces to the corresponding statement for perfect complexes on Noetherian schemes.

Proposition 0.13. *Suppose that $\mathcal{S}$ is a Noetherian concentrated algebraic stack. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be flat concentrated morphisms. Consider the hypotheses of Lemma 0.9. Suppose $K \in D^b_{\mathrm{coh}}(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})$. Assume that $\mathbf{R}f_* \mathbf{R}\mathcal{H}om(G, G)$ and $\mathbf{R}g_* \mathbf{R}\mathcal{H}om(\Phi_K(G), \Phi_K(G))$ are perfect on $\mathcal{S}$ (e.g. this happens if f, g are proper). Set $C_\phi := \mathrm{cone}(\phi)$ where ϕ is the morphism in (0.4). Then:*

(1) *The base changed integral transform Φ_{K_t} at $p \in |\mathcal{S}|$ being fully faithful is independent of the choice of representative $t\colon \mathrm{Spec}(k) \to \mathcal{S}$ of p.*
(2) *The set of points as in (1) equals $|\mathcal{S}| \setminus \mathrm{Supp}(C_\phi)$. Therefore, this locus is open.*

Proof. Independence of the representative follows from Lemma 0.12. Indeed, any two field valued representatives of the same point admit a common field extension. Full faithfulness ascends to the common extension by (1), since spectra of fields admit compact generators, and descends to the other representative by (2).

Fix a representative $t\colon \mathrm{Spec}(k) \to \mathcal{S}$ of a point p. Since f and g are flat, Lemma 0.10 identifies $\mathbf{L}t^*\phi$ with the comparison morphism

$$\phi_t : \mathbf{R}(f_t)_* \mathbf{R}\mathcal{H}om(G_t, G_t) \to \mathbf{R}(g_t)_* \mathbf{R}\mathcal{H}om(\Phi_{K_t}(G_t), \Phi_{K_t}(G_t)).$$

Hence, $\mathbf{L}t^*C_\phi \cong \operatorname{cone}(\phi_t)$.

The object G_t is a compact generator of $D_{\mathrm{qc}}(\mathcal{X}_t)$. Moreover, $\Phi_{K_t}(G_t) \cong \mathbf{L}b_t^*\Phi_K(G)$ is perfect where $b_t\colon \mathcal{Y}_t \to \mathcal{Y}$ the natural projection. Thus, as in the proof of Lemma 0.12, Φ_{K_t} takes compact objects to compact objects and admits a right adjoint. Consequently, Lemma 0.9, applied over $\operatorname{Spec}(k)$, gives

$$\Phi_{K_t} \text{ is fully faithful} \iff \phi_t \text{ is an isomorphism} \iff \mathbf{L}t^*C_\phi \cong 0.$$

By hypothesis, the source and target of ϕ are perfect, and hence, C_ϕ is perfect. Hence, its cohomological support is closed in $|\mathcal{S}|$, and

$$\mathbf{L}t^*C_\phi \cong 0 \iff p \notin \operatorname{Supp}(C_\phi).$$

Therefore, the full faithfulness locus is $|\mathcal{S}| \setminus \operatorname{Supp}(C_\phi)$, which is open. □

Remark 0.14. The proof of Proposition 0.13 is not scheme-theoretic. It applies to Noetherian algebraic stacks satisfying the stated concentratedness, compact generation, perfectness, and base change hypotheses. In particular, the openness assertion is a consequence of the full faithfulness criterion and its compatibility with base change. The argument also illustrates a distinction relevant to the equivalence locus. The full faithfulness criterion above can be transported by ordinary pullback-pushforward base change and does not require base change for a right adjoint to Φ_K. By contrast, a counit based criterion for equivalence requires comparison of the right adjoint with the right adjoint after base change, together with compatibility of the corresponding counits.

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