OPENNESS OF FOURIER–MUKAI LOCI FOR ALGEBRAIC SPACES

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ABSTRACT. We develop a theory of relative perfect complexes on algebraic spaces. Our main result establishes openness for the loci where fibers of an integral transform are fully faithful (resp. an equivalence), and we obtain new examples in arbitrary characteristic. In fact, we explicitly identify the complements of these loci.

CONTENTS

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1. Introduction

Integral transforms have become one of the fundamental tools for studying derived categories of algebraic varieties. This article addresses the following natural question:

Question 1.1. How does Fourier–Mukai partnership vary with points on the base?

Let us make this more concrete. Suppose $f_i \colon Y_i \to S$ are proper flat morphisms of Noetherian algebraic spaces. Consider the fibered square

$$\begin{tikzcd} Y_1 \times_S Y_2 \arrow[r, "f_1'"] \arrow[d, "f_2'"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & S. \end{tikzcd}$$

For any $E \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$, the *integral transform with kernel E* is the functor $\Phi_E \colon D_{\mathrm{qc}}(Y_1) \to D_{\mathrm{qc}}(Y_2)$ given by the assignment $A \mapsto \mathbf{R}(f_1')_*(\mathbf{L}(f_2')^* A \otimes^{\mathbf{L}} E)$. If Φ_E yields a derived equivalence, then we say that Y_1 and Y_2 are *Fourier–Mukai S-partners*. Given $p \in |S|$, a representative $t \colon \operatorname{Spec}(k) \to S$ of p, and $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$, let Φ_{K_p} be the fiber of the integral transform $D_{\mathrm{qc}}(Y_1 \times_S \operatorname{Spec}(k)) \to D_{\mathrm{qc}}(Y_2 \times_S \operatorname{Spec}(k))$. In particular, $\Phi_{K_p} := \Phi_{\mathbf{L}(t')^* K}$ where $t' \colon Y_1 \times_S Y_2 \times_S \operatorname{Spec}(k) \to Y_1 \times_S Y_2$ is the natural morphism. Define

$$\mathrm{fm}(K) := \{p \in |S| : \Phi_{K_p} \text{ is fully faithful}\}$$

and

$$\mathrm{FM}(K) := \{p \in |S| : \Phi_{K_p} \text{ is an equivalence}\}.$$

This is independent of the representative for p (see Lemma 5.7). We want to address whether or not these loci are open. In doing so, we show that equivalences and fully faithfulness of integral transforms are stable under generalizations.

1.1. What is known.

Although derived categories for algebraic spaces are now reasonably better understood, there are no effective tools for studying Question 1.1 in the presence of singularities. A major obstacle is that existing techniques for studying integral transforms typically require the kernel to be perfect. See e.g. [HP26, Theorem 5.8] and [BS20, Proposition 4.9]. In particular, the arguments of [AL12] do not extend to nonperfect complexes. Such perfectness assumptions are too restrictive. See Remark 4.21 for details.

This limitation already appears in the study of base change. A classical result of Orlov shows that an integral transform inducing an equivalence between smooth projective varieties remains an equivalence after extension of the ground field [Orl02, Lemma 2.12]. Recently, [GLMP25] established an analog for singular varieties.

However, the proofs are considerably different. Orlov's proof relies on morphisms between perfect complexes representing the unit and counit of the relevant adjunctions, whereas such morphisms need not exist for nonperfect kernels. Consequently, the techniques developed for perfect kernels do not extend to the singular setting.

The following illustrates that nonperfect kernels are unavoidable.

Example 1.2. Let $k := \mathbb{F}_3(t)$ where t is transcendental. Consider the projective plane curve $f \colon C \to \operatorname{Spec}(k)$ given by the equation $y^2 z + x^3 - tz^3 = 0$. This is a regular Noetherian scheme as it is normal and of Krull dimension one. However, after base change of C along the field extension $\ell := \mathbb{F}_3(t^{\frac{1}{3}})$, $C \times_k \operatorname{Spec}(\ell)$ is a singular projective curve [Kol11, Remark 16]. Denote by t' the natural base change morphism

$$C \times_k C \times_k \operatorname{Spec}(\ell) \to C \times_k C.$$

Choose any $K \in D^b_{\mathrm{coh}}(C \times_k C)$ for which its associated integral transform $\Phi_K \colon D_{\mathrm{qc}}(C) \to D_{\mathrm{qc}}(C)$ is an equivalence. For example, we can take $K := (\Delta_f)_*\mathcal{O}_C$, which cannot be perfect (see e.g. [DLM24, Proposition 4.8], and use that C is not smooth). Such a kernel K in general cannot be perfect. Indeed, if K were perfect, then $\mathbf{L}(t')^*K$ is too. By [GLMP25, Corollary 1.3], the associated integral transform remains an equivalence after base change. However, [DLM25, Proposition 4.6] implies that $\mathbf{L}(t')^*K$ cannot be perfect because $C \times_k \ell$ is singular, which is a contradiction.

Example 1.2 shows that any theory of Fourier–Mukai transforms capable of addressing Question 1.1 must allow nonperfect kernels of integral transforms. Recent work has started developing a theory for schemes [GLMP25]. However, it is not powerful enough to address Question 1.1. The purpose of this article is to develop such a theory for algebraic spaces.

1.2. **What we do**. The main contribution of this article is a framework allowing Fourier–Mukai theory to be developed for pseudocoherent kernels on algebraic spaces. This is discussed in detail in §1.3.

1.2.1. ***Fourier–Mukai locus***. Our main result is an openness property for the locus of points where fully faithfulness or equivalences occur at fibers.

Theorem 1.3. *Consider proper flat surjective morphisms $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ of Noetherian algebraic spaces. For any $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ which is relatively perfect over each Y_i, $\mathrm{fm}(K)$ and $\mathrm{FM}(K)$ are open subsets of $|S|$.*

See Theorem 5.8. Even for schemes, this appears to be new, particularly because it applies uniformly in arbitrary characteristic, including mixed characteristic. Motivated by this theorem, we introduce the notion of the Fourier–Mukai locus (Definition 5.6). As an example, let S be a Dedekind scheme and $Y \to S$ an elliptic fibration extending an elliptic curve over the generic point. Since the generic fiber admits a Fourier–Mukai autoequivalence, openness implies that all but finitely many closed fibers do as well (i.e. derived equivalences for all but finitely many reductions modulo primes). See Example 5.9.

The proof identifies the complement of $\mathrm{fm}(K)$ and $\mathrm{FM}(K)$ with the image of the support of the cones of the unit or counit of an adjunction applied to compact generators. Specifically, fully faithfulness is controlled by the support of the cone of the unit on a compact generator, whereas equivalence is controlled by the union of the supports of the cones of the unit and counit for compact generators. See the proof of Theorem 5.8 for details. Neither this description nor the resulting openness theorem appears to be known even for schemes. This generalizes the algebraic space version of [HP26, Corollary 5.9] in two ways: first, we do not require perfect kernels; second, the arguments differ since that of [AL12] are not available. In fact, the techniques here are independent.

1.2.2. ***Autoequivalences***. We start by studying endofunctors on elliptic fibrations of algebraic spaces. Recall that an *elliptic X-fibration* is a proper Gorenstein morphism $Y \to X$ between Noetherian algebraic spaces whose geometric fibers are integral curves of arithmetic genus one with trivial dualizing sheaf (see Reminder 5.2 for precise formulation). Basic examples include fiber products of (perhaps singular) elliptic curves (see Example 5.3).

It turns out that the ideal sheaf for an elliptic fibration yields an autoequivalence on D^b_{coh}. This includes the case of Noetherian algebraic spaces of mixed characteristic. See Theorem 5.5. A key ingredient is a study of the behavior of diagonal ideal sheaves under affine base change (Proposition 5.4), allowing us to generalize [HCS09, Proposition 2.16] to algebraic spaces. These results are useful for examples later (see §5.2).

1.3. **A framework.** A guiding principle throughout this article to address Question 1.1 is that global categorical behavior—such as equivalences, fully faithful functors, and adjunctions—is controlled by fiberwise data. This philosophy already appears in [HCS09] where it is shown that, for locally projective flat morphisms of finitely presented schemes over an algebraically closed field, fully faithfulness and equivalences of integral transforms may be detected on special fibers. Recently, [GLMP25] extended this principle to proper flat morphisms of Noetherian schemes.

Our technical results extend these fiberwise criteria to algebraic spaces. Doing so requires substantial new technical developments, which we now describe. The theory developed concerns the behavior of integral transforms along base change, compatibility with adjunctions, and behavior along fibers controlling derived equivalences and fully faithfulness.

All derived categories in this article are formed on small étale sites. Several inputs in the literature are stated on small Zariski or lisse–étale sites. We therefore record the arguments needed to transport them to our setting, rather than appeal informally to reductions to schemes.

1.3.1. ***Relative perfection.*** Various notions of relative perfectness have been studied for schemes [Ill71, AJS23, HCS09, Bal09, Riz17]. These notions characterize when an integral transform preserves bounded pseudocoherent or perfect complexes. We prove that the various definitions of relative perfectness appearing throughout the literature are equivalent. See Proposition 3.16. This comparison appears to be missing from the literature. See e.g. the discussion in [AJS23, §1.4] and the remarks following [HCS07, Definition 1.3].

Our work introduces analogous notions for algebraic spaces: *relative perfectness* and *relative quasi-perfectness*. See §§3.3 and 3.4 for details. The first notion is motivated by [AJS23], whereas the second is inspired by [Bal09, §3]. In general, the two notions do not coincide; see Example 3.28.

In the case of proper morphisms of algebraic spaces, we show that relative perfectness coincides with relative quasi-perfectness. See Theorem 3.43. These notions allow us to determine precisely when integral transforms preserve complexes in D^b_{coh} or Perf. See Corollary 3.45. A key technical ingredient for both results is a characterization of perfect complexes on algebraic spaces, generalizing [AJS23, Theorem 2.3]. See Proposition 3.12.

The implication from relative perfectness to preservation of bounded pseudocoherent complexes relies on detecting boundedness of cohomology via the vanishing of Ext-groups against a compact generator. See Lemma 3.38.

The main difficulty lies in proving that relative quasi-perfectness coincides with relative perfectness. Our arguments combine compact generation of triangulated categories with generation by thick subcategories in the sense of [BV03]. A key observation is that relatively quasi-perfect complexes admit natural involutions that induce useful adjunctions. In fact, these adjunctions drive the comparison between the two notions. See §3.5 for details.

1.3.2. ***Base change behavior.*** Grothendieck duality and compatibility with base change lie at the heart of our approach. The central result of this section is to prove a fiberwise criterion. In particular, we show that an integral transform is fully faithful or an equivalence on D^b_{coh} is completely determined by its behavior on special geometric fibers of the base. See Theorem 4.34. The result is a space theoretic analog of [GLMP25, Theorem 1.2], but its proof requires many new ideas.

Unlike schemes, the underlying topological space of an algebraic space consists of equivalence classes of morphisms from spectra of fields. Hence, one must establish compatibility with faithfully flat and affine base change, as well as independence of the chosen representative of a point.

The first ingredient is a study of relative perfectness under base change. We prove for pseudocoherent complexes that relative perfectness can be detected after faithfully flat presentations and on fibers over closed points. See Theorem 4.9. This provides the descent and ascent results needed throughout.

The second ingredient is the construction of adjoints for integral transforms, and the proof of their compatibility with base change. Here we rely on Neeman's extension of Grothendieck duality to algebraic spaces [Nee23]. See Appendix C for details. The key new tool is a theory of derived reflexivity for algebraic spaces, which extends [AIL11].

In particular, we show that derived reflexivity is étale local. See Lemma 4.13. Also, we prove that relatively perfect complexes are naturally reflexive with respect to 'relative dualizing complexes'. See Lemma 4.15. These results allow us to construct explicit adjoints for integral transforms and establish their compatibility with base change.

With these ingredients, the arguments proving the fiberwise criterion largely parallel those for schemes in [GLMP25, §4]. The proof rests on three ingredients:

- existence and control of adjoints (Proposition 4.17 and Corollary 4.18);
- compatibility of integral transforms with flat base change (Lemmas 4.4 and 4.22);
- detection of relative perfectness and boundedness by faithfully flat descent (Theorem 4.9).

The results above recover space theoretic analogs of recent ascent and descent theorems for schemes [DLM25, GLMP25]. Also, they extend results on adjoints of integral transforms due to [Bal09, Riz17]. The key results needed are Theorem 4.9 and Theorem 4.34. See §4 for details. As an application of our methods, we also obtain a new proof of [GLMP25, Proposition 4.9] (and hence, [HCS09, Proposition 2.15]), making our approach largely self-contained.

We compare our techniques to the literature. Existing proofs for perfect kernels ultimately reduce full faithfulness and equivalence to verifying that certain morphisms between perfect complexes are isomorphisms [HP26, Theorem 5.8]. This strategy follows [AL12] and fundamentally relies on perfectness (see Remark 4.21). Since these morphisms need not exist for pseudocoherent kernels, a different approach is required.

Our approach replaces morphisms between perfect kernels by arguments based on mates together with the machinery developed in §§3 and 4. This differs greatly than the use of mates in [BS20] for studying integral transforms. Their arguments rely crucially on kernels being perfect, whereas our kernels need only be pseudocoherent. In particular, the techniques of [BS20, Proposition 4.9] are unavailable, and the machinery developed in the preceding sections is essential.

More broadly, the techniques developed here provide a foundation for studying integral transforms on algebraic spaces without perfectness assumptions, opening the way to further applications in moduli theory and singularity theory.

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2. Preliminaries

2.1. **Generation**. We discuss a form of generation for triangulated categories. See [BV03] for details. Let $\mathcal{T}$ be a triangulated category with shift functor $[1]\colon \mathcal{T} \to \mathcal{T}$. Consider a subcategory $S \subseteq \mathcal{T}$. A triangulated subcategory of $\mathcal{T}$ is called **thick** if it is closed under direct summands. Denote by $\langle S \rangle$ the smallest thick subcategory of $\mathcal{T}$ containing S. If S consists of a single object G, we write $\langle S \rangle := \langle G \rangle$. Set $\operatorname{add}(S)$ to be the smallest strictly full subcategory of $\mathcal{T}$ containing S that is closed under shifts, finite coproducts, and direct summands. Inductively, let $\langle S \rangle_0$ consist of all zero objects in $\mathcal{T}$, $\langle S \rangle_1 := \operatorname{add}(S)$, and

$$\langle S \rangle_n := \operatorname{add}\{\operatorname{cone}(\phi) \mid \phi \in \operatorname{Hom}_{\mathcal{T}}(\langle S \rangle_{n-1}, \langle S \rangle_1)\}.$$

It can be checked that $\langle S \rangle = \cup_{n=0}^{\infty} \langle S \rangle_n$. We say E is **finitely built by** S if $E \in \langle S \rangle$.

Example 2.1. Let $X = \operatorname{Spec}(R)$ where R is a regular ring. Then $\langle \mathcal{O}_X \rangle = D^b_{\operatorname{coh}}(X)$. More generally, let X be an affine Noetherian scheme and $Z \subseteq X$ be closed. If $P \in \operatorname{Perf}(X)$ satisfies $\operatorname{supp}(P) = Z$, then $\langle P \rangle = \operatorname{Perf}(X) \cap D^b_{\operatorname{coh},Z}(X)$. See [Nee92, Lemma 1.2].

If $\mathcal{T}$ admits small coproducts, then the collection of compact objects in $\mathcal{T}$ is denoted by $\mathcal{T}^c$. These form a triangulated subcategory of $\mathcal{T}$. We say that $\mathcal{T}$ is **compactly generated** if it coincides with the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{T}^c$ and closed under small coproducts. Equivalently, $\mathcal{T}$ is compactly generated if, for any $E \in \mathcal{T}$ satisfying $\operatorname{Hom}(P,E) = 0$ for all $P \in \mathcal{T}^c$, one has $E \cong 0$ [SS03, Lemma 2.2.1]. Note that classical generators for $\mathcal{T}^c$ coincide with compact generators for $\mathcal{T}$ [Sta26, Tag 09SR].

Example 2.2. Let X be a quasi-compact quasi-separated scheme. Then $D_{\operatorname{qc}}(X)^c = \operatorname{Perf}(X)$, and moreover, $\operatorname{Perf}(X)$ admits a classical generator. See [BV03, Theorem 3.1.1]. More generally, let $Z \subseteq X$ be closed with quasi-compact complement. By [Rou08, Theorem 6.8], $D_{\operatorname{qc},Z}(X)$ is compactly generated by a single object. Hence, $\operatorname{Perf}(X) \cap D_{\operatorname{qc},Z}(X)$ admits a classical generator.

2.2. **t-structures**. We discuss t-structures on a triangulated category $\mathcal{T}$ and refer to [KV88, BBDG18]. A pair of strictly full subcategories $\tau = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of $\mathcal{T}$ is a **t-structure** if:

- $\operatorname{Hom}(A,B) = 0$ for all $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}[-1]$,
- $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}[-1] \subseteq \mathcal{T}^{\geq 0}$,
- for every $E \in \mathcal{T}$, there is a distinguished triangle
$$\tau^{\leq 0}E \to E \to \tau^{\geq 1}E \to (\tau^{\leq 0}E)[1]$$
with $\tau^{\leq 0}E \in \mathcal{T}^{\leq 0}$ and $\tau^{\geq 1}E \in \mathcal{T}^{\geq 0}[-1]$.

The above distinguished triangle is unique up to unique isomorphism, and it is called the **truncation triangle** of E with respect to τ. Given $n \in \mathbb{Z}$, the pair $(\mathcal{T}^{\leq n}, \mathcal{T}^{\geq n})$ is also a t-structure on $\mathcal{T}$ where $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$. A pair of t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ are **equivalent** if there exists an $n \geq 0$ such that $\mathcal{T}_2^{\leq -n} \subseteq \mathcal{T}_1^{\leq 0} \subseteq \mathcal{T}_2^{\leq n}$. Given a set $\mathcal{S} \subseteq \mathcal{T}$ where $\mathcal{T}$ is well-generated, we let $\overline{\langle \mathcal{S} \rangle}^{(-\infty,0]}$ be the smallest cocomplete aisle of $\mathcal{T}$ containing $\mathcal{S}$. By [Nee21, Theorem 2.3], $\overline{\langle \mathcal{S} \rangle}^{(-\infty,0]}$ always exists.

Example 2.3. Assume that $\mathcal{T}$ admits small coproducts. Let $\mathcal{A}$ be a full subcategory of $\mathcal{T}^c$ closed under positive shifts. Denote by $\operatorname{Coprod}(\mathcal{A})$ the smallest strictly full subcategory of $\mathcal{T}$ that contains $\mathcal{A}$ which is closed under extensions and small coproducts. By [CHNS24, Theorem 2.3.3 & Remark 2.3.4] (which generalizes [ATLS03, Theorem A.1 & Proposition A.2]), this construction defines an aisle in $\mathcal{T}$. We call the associated t-structure $\tau_{\mathcal{A}}$ the ***t*-structure compactly generated by** $\mathcal{A}$. If $\mathcal{A} = \{G[i] \mid i \geq 0\}$ for some compact object G; we denote the corresponding compactly generated t-structure by τ_G. If $\mathcal{T}$ is compactly generated by a single object G, we define the **preferred equivalence class** to be the equivalence class of t-structures containing the t-structure compactly generated by G.

Let $F \colon \mathcal{T}_1 \to \mathcal{T}_2$ be an exact functor between triangulated categories equipped with t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$. We say that F is **right *t*-exact** if $F(\mathcal{T}_1^{\leq 0}) \subseteq \mathcal{T}_2^{\leq 0}$, and **left *t*-exact** if $F(\mathcal{T}_1^{\geq 0}) \subseteq \mathcal{T}_2^{\geq 0}$. If both conditions hold, then F is ***t*-exact**. We say τ is **nondegenerate** if $\cap_{n\in\mathbb{Z}} \mathcal{T}^{\geq n} = \cap_{n\in\mathbb{Z}} \mathcal{T}^{\leq n} = \operatorname{add}(0)$ (i.e. consists of only zero objects). Moreover, τ is called **bounded** if for every $E \in \mathcal{T}$ there exists an $n \geq 0$ such that $E[n] \in \mathcal{T}^{\leq 0}$ and $E[-n] \in \mathcal{T}^{\geq 0}$.

2.3. **Algebraic spaces**. We follow [Sta26] for conventions on algebraic spaces. In this subsection, let X be a quasi-compact quasi-separated algebraic space.

2.3.1. *Notions.* An **étale presentation** of X is an étale, finitely presented, surjective morphism to X from a scheme. The underlying topological space of X is given by equivalence classes of morphisms from fields to the stack [Sta26, Tag 03BT]. We denote it by $|X|$. Recall that a morphism from a field to a quasi-separated algebraic space is an affine morphism [Sta26, Tag 09TF].

2.3.2. *Categories.* $\operatorname{Mod}(X)$ is the Grothendieck abelian category of sheaves of $\mathcal{O}_X$-modules on the small étale site $X_{\text{étale}}$ of X [Sta26, Tag 03LT]. $\operatorname{Qcoh}(X)$ is the full subcategory of $\operatorname{Mod}(X)$ consisting of quasi-coherent sheaves. $D(X) := D(\operatorname{Mod}(X))$ is the derived category of $\operatorname{Mod}(X)$. $D_{\text{qc}}(X)$ is the full subcategory of $D(X)$ consisting of complexes with quasi-coherent cohomology sheaves. $\operatorname{Perf}(X)$ is the full subcategory of perfect complexes in $D_{\text{qc}}(X)$. If X is Noetherian, then $\operatorname{coh}(X)$ is the full subcategory of $\operatorname{Mod}(X)$ consisting of coherent sheaves and $D^b_{\text{coh}}(X)$ denotes the full subcategory of $D(X)$ consisting of bounded pseudocoherent complexes.

2.3.3. *Support.* For any $M \in \operatorname{Qcoh}(X)$, define $\operatorname{supp}(M) := p\big(\operatorname{supp}(p^*M)\big) \subseteq |X|$ where $p \colon U \to X$ is any étale presentation from a scheme; this is independent of p [Sta26, Tag 07TY]. This coincides with the notion of support for abelian sheaves on the small étale site [Sta26, Tag 04K7]. For $E \in D_{qc}(X)$, set

$$\operatorname{supp}(E) := \bigcup_{j \in \mathbb{Z}} \operatorname{supp}\left(\mathcal{H}^j(E)\right) \subseteq |X|.$$

Given a closed subset $Z \subseteq |X|$, we say E is **supported on** Z if $\operatorname{supp}(E) \subseteq Z$. If $E \in \operatorname{Qcoh}(X)$ and is of finite type, then $\operatorname{supp}(E) \subseteq |X|$ is closed [Sta26, Tag 07TZ]. On a locally Noetherian algebraic space, quasi-coherent sheaves of finite type coincide with coherent sheaves [Sta26, Tag 07UB]. Denote by $D_{qc,Z}(X)$ the strictly full subcategory of $D_{qc}(X)$ consisting of objects supported on Z.

Lemma 2.4. *Let X be a decent algebraic space (e.g. quasi-separated [Sta26, Tag 03I7]). If $t\colon \operatorname{Spec}(k) \to X$ is a morphism from a field such that $t(\operatorname{Spec}(k)) = p$, then t represents p (see [Sta26, Tag 03BT]).*

Proof. Let $i\colon Z_p \to X$ be the residual space of X at p (see [Sta26, Tags 06QZ & 06R0]). Choose any $h\colon \operatorname{Spec}(\ell) \to X$ that represents p. By [Sta26, Tag 06QZ] (e.g. use X is decent), there exists a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(k) \arrow[r, "t'"] \arrow[dr, "t"'] & Z_p \arrow[d, "i"] \\ & X. \end{tikzcd}$$

Moreover, from [Sta26, Tag 0H1T], there exists a commutative diagram

$$\begin{tikzcd} Z_p \arrow[d, "i"'] & \operatorname{Spec}(\ell) \arrow[l, "h'"'] \arrow[dl, "h"] \\ X. & \end{tikzcd}$$

Consider the fiber product

$$\begin{tikzcd} \operatorname{Spec}(k) \times_{Z_p} \operatorname{Spec}(\ell) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \operatorname{Spec}(\ell) \arrow[d, "h'"] \\ \operatorname{Spec}(k) \arrow[r, "t'"'] & Z_p. \end{tikzcd}$$

Since $|Z_p|$ is a singleton, t' is surjective. Hence, by base change, p_2 is surjective. Thus, $\operatorname{Spec}(k) \times_{Z_p} \operatorname{Spec}(\ell)$ is nonempty. Here $\operatorname{Spec}(k) \times_{Z_p} \operatorname{Spec}(\ell)$ is an algebraic space. Choose an étale presentation $s\colon U \to \operatorname{Spec}(k) \times_{Z_p} \operatorname{Spec}(\ell)$. Fix some $p' \in |\operatorname{Spec}(k) \times_{Z_p} \operatorname{Spec}(\ell)|$. Consider any $g\colon \operatorname{Spec}(K) \to \operatorname{Spec}(k) \times_{Z_p} \operatorname{Spec}(\ell)$ that represents p'. Then we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "p_2 \circ g"] \arrow[d, "p_1 \circ g"'] & \operatorname{Spec}(\ell) \arrow[d, "h"] \\ \operatorname{Spec}(k) \arrow[r, "t"'] & X. \end{tikzcd}$$

This completes the proof. ◻

Lemma 2.5. *Let X be a Noetherian algebraic space and $E \in D_{\mathrm{qc}}(X)$ have coherent cohomology. Then $\operatorname{supp}(E)$ coincides with the $p \in |X|$ such that $\mathbf{L}i^*E \ncong 0$ where i is a representative of p; in fact, this is independent of the representative.*

Proof. Let $s\colon U \to X$ be an étale presentation. The first claim follows from [Sta26, Tag 07TZ] and the string of equalities:

$$\begin{aligned}
\operatorname{supp}(E) &= \bigcup_n \operatorname{supp}(\mathcal{H}^n(E)) \\
&= \bigcup_n s(\operatorname{supp}(s^*\mathcal{H}^n(E))) \\
&= \bigcup_n s(\operatorname{supp}(\mathcal{H}^n(s^*E))) \\
&= \bigcup_n s(\operatorname{supp}(\mathcal{H}^n(\mathbf{L}s^*E))) \\
&= s(\bigcup_n \operatorname{supp}(\mathcal{H}^n(\mathbf{L}s^*E))) \\
&= s(\operatorname{supp}(\mathbf{L}s^*E)).
\end{aligned}$$

By [Sta26, Tag 056J], $\operatorname{supp}(s^*\mathcal{H}^n(E))$ is the set of $q \in U$ such that $i^*s^*\mathcal{H}^n(E)$ is nonzero where $i\colon \operatorname{Spec}(\kappa(q)) \to U$ is the natural morphism. Choose $p \in X$ such that $\mathbf{L}i^*E \ncong 0$ where i is a representative of p. Since s is surjective, Lemma 2.4 allows us to choose a representative i' of some $q \in s^{-1}(p)$. We claim that $\mathbf{L}(s \circ i')^*E \ncong 0$. Indeed, there exists a field ℓ and commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(\ell) \arrow[r, "g"] \arrow[d, "h"'] & \operatorname{Spec}(k) \arrow[d, "i"] \\
\operatorname{Spec}(\kappa(q)) \arrow[r, "s\circ i'"'] & X.
\end{tikzcd}$$

If $\mathbf{L}i^*E \ncong 0$, then $\mathbf{L}(i \circ g)^*E \ncong 0$. Hence, by Lemma 2.10, $\mathbf{L}(s \circ i')^*E$ is nonzero. Notice that this shows independence of the choice of representative for p regarding whether the derived pullback of E is nonzero. Now, from [ILN15, Lemma A.1], $q \in \operatorname{supp}(\mathbf{L}s^*E)$, and so, $p \in \operatorname{supp}(E)$. We check the reverse inclusion. Choose $p \in \operatorname{supp}(E)$. As s is surjective, we can find a representative $\operatorname{Spec}(\kappa(q)) \to X$ given by the composition of s with the natural morphism $i\colon \operatorname{Spec}(\kappa(q)) \to U$ for some $q \in \operatorname{supp}(\mathbf{L}s^*E)$. By [ILN15, Lemma A.1], we have that $\mathbf{L}(s \circ i)^*E \ncong 0$, and so we are done. Observe that [ILN15] proves results on the small Zariski site of U. However, it is valid for the small étale site of U. Indeed, we can apply [Sta26, Tag 071Q] to realize that the morphism of ringed topoi $\epsilon\colon (U_{\text{étale}}, \mathcal{O}_{U_{\text{étale}}}) \to (U_{\mathrm{Zar}}, \mathcal{O}_{\mathrm{Zar}})$ yields a t-exact equivalence

$$\mathbf{L}\epsilon^*\colon D_{\mathrm{qc}}(\mathcal{O}_{\mathrm{Zar}}) \leftrightarrows D_{\mathrm{qc}}(\mathcal{O}_{\text{étale}})\colon \mathbf{R}\epsilon_*.$$

By [Sta26, Tag 07TY], the support of an abelian sheaf on $U_{\text{étale}}$ agrees with the notion as defined for quasi-coherent modules on schemes. This completes the proof. □

2.3.4. *Perfect complexes.* Perfect complexes are defined on any ringed site [Sta26, Tag 08G4], e.g. on the small étale site of X. A complex is **strictly perfect** if it is a bounded complex whose terms are direct summands of finite free modules. A complex is **perfect** if it is locally strictly perfect. For any $Z \subset |X|$ closed on a quasi-compact quasi-separated algebraic space such that $|X| \setminus Z$ is quasi-compact, perfect complexes of $D_{\mathrm{qc},Z}(X)$ coincide with compacts of $D_{\mathrm{qc},Z}(X)$, and in particular they are perfect complexes in $D_{\mathrm{qc}}(X)$. Moreover, $D_{\mathrm{qc},Z}(X)$ is singly compactly generated for such Z. See [Sta26, Tags 0AED, 09M8, & 0AEC].

2.3.5. *Internal homs.* Let $E, G \in D(X)$. There exists the derived tensor product $E \otimes^{\mathbf{L}} G \in D(X)$ and derived sheaf Hom $\mathbb{R}\mathcal{H}om(E,G) \in D(X)$. Here, $(-) \otimes^{\mathbf{L}} E$ is left adjoint to $\mathbb{R}\mathcal{H}om(E,-)$ on $D(X)$. Applying [Sta26, Tags 08F5 & 0FPQ], D_{qc} is symmetric monoidal, i.e. $E \otimes^{\mathbf{L}} G \in D_{\mathrm{qc}}(X)$ for all $E, G \in D_{\mathrm{qc}}(X)$. However, this need not be the case for $\mathbb{R}\mathcal{H}om(E,G)$ despite the formation of $\mathbb{R}\mathcal{H}om(E,-)$ being étale local [Sta26, Tags 04LX & 08JB]. By [Sta26, Tag 09IY], the category $D_{\mathrm{qc}}(X)$ is singly compactly generated. Also, the endofunctor $(-) \otimes^{\mathbf{L}} E$ on $D_{\mathrm{qc}}(X)$ preserves small coproducts. Hence, [Nee01, Theorem 8.4.4] implies that the endofunctor admits a right adjoint $\mathbf{R}\mathcal{H}om(E,-)$ on $D_{\mathrm{qc}}(X)$. This equips $D_{\mathrm{qc}}(X)$ with the structure of a closed symmetric monoidal triangulated category. Denote by $i \colon D_{\mathrm{qc}}(X) \to D(X)$ for the natural inclusion. It admits a right adjoint $Q \colon D(X) \to D_{\mathrm{qc}}(X)$ by reasoning above because i preserves small coproducts.

Lemma 2.6. *Let X be an algebraic space. If K is perfect and $L, M \in D(X)$, then there exists a functorial isomorphism*

$$K \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(M,L) \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(K,M),L).$$

Proof. We compute this directly:

$$\begin{aligned} K \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(M,L) &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(K,\mathcal{O}_X),\mathbb{R}\mathcal{H}om(M,L)) && (\text{[Sta26, Tag 08JJ]}) \\ &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(K,\mathcal{O}_X) \otimes^{\mathbf{L}} M, L) && (\text{[Sta26, Tag 08J9]}) \\ &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(K,M),L) && (\text{[Sta26, Tag 08JJ]}). \end{aligned}$$

□

Lemma 2.7. *Let X be an algebraic space. For any $E, L \in D_{\mathrm{qc}}(X)$, there is a natural isomorphism*

$$\mathbf{R}\mathcal{H}om(E,L) \to Q(\mathbb{R}\mathcal{H}om(i(E),i(L))).$$

In particular, if $\mathbb{R}\mathcal{H}om(i(E),i(L))$ has quasi-coherent cohomology, then

$$\mathbf{R}\mathcal{H}om(i(E),i(L)) \cong \mathbb{R}\mathcal{H}om(i(E),i(L)).$$

Proof. This is well-known but we clarify it on the small étale site. Note that there exist adjunctions

$$\begin{tikzcd} D_{\mathrm{qc}}(X) \arrow[r, shift left, "i"] & D(X) \arrow[l, shift left, "Q"] \arrow[r, shift left, "((-)\otimes^{\mathbf{L}} i(E))"] & D(X). \arrow[l, shift left, "\mathbb{R}\mathcal{H}om(i(E)-)"] \end{tikzcd}$$

By composing, we obtain

$$\begin{tikzcd} D_{\mathrm{qc}}(X) \arrow[r, shift left, "((-)\otimes^{\mathbf{L}} i(E))\circ i"] & D(X). \arrow[l, shift left, "Q(\mathbb{R}\mathcal{H}om(i(E),-))"] \end{tikzcd}$$

Using that i is monoidal, it follows that $((-)\otimes^{\mathbf{L}} i(E))\circ i$ induces an endofunctor on $D_{\mathrm{qc}}(X)$, whose right adjoint is $\mathbf{R}\mathcal{H}om(E,-)$. Therefore, restriction gives an adjunction

$$D_{\mathrm{qc}}(X) \underset{Q(\mathbb{R}\mathcal{H}om(i(E),-))}{\overset{((-)\otimes^{\mathbf{L}} i(E))\circ i}{\rightleftarrows}} D_{\mathrm{qc}}(X).$$

Thus, the desired claim follows from uniqueness of adjoints. □

2.3.6. *Functors.* Let $f\colon Y\to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Recall that there exists an adjoint pair

$$\mathbf{L}f^*\colon D(X)\leftrightarrows D(Y)\colon \mathbf{R}f_*.$$

See [Sta26, Tags 07A6]. By [Sta26, Tags 08FA & 08F4], $\mathbf{L}f^*$ and $\mathbf{R}f_*$ preserve quasi-coherent cohomology. Also, from [Sta26, Tag 07A6], the restrictions of $\mathbf{L}f^*$ and $\mathbf{R}f_*$ form an adjoint pair on D_{qc}. Using [Sta26, Tag 07A4], it follows that $\mathbf{L}f^*$ is monoidal. Lastly, $\mathbf{R}f_*$ admits a right adjoint $f^\times$ [Sta26, Tags 0E55].

2.3.7. *Adjoint pseudofunctors.* We need the following formalism. See [LH09, §3.6] for details.

Lemma 2.8. *Let* Sp *be the category of quasi-compact quasi-separated algebraic spaces over a base scheme (e.g. $\mathbb{Z}$). For each $X\in$ Sp, set $X^*=X_*=D_{\mathrm{qc}}(X)$, which is a closed Δ-category with product $\otimes$ and internal Hom $\mathbf{R}\mathcal{H}om(E,-)$. For $g\colon W\to Y$ and $f\colon Y\to X$ in* Sp*, write $f^*:=\mathbf{L}f^*\colon D_{\mathrm{qc}}(X)\to D_{\mathrm{qc}}(Y)$ and $f_*:=\mathbf{R}f_*\colon D_{\mathrm{qc}}(Y)\to D_{\mathrm{qc}}(X)$, and $d_{f,g}$ and $c_{f,g}$ be respectively as in* [Sta26, Tags 0D6D & 0D6E]. *In the sense of* [LH09, (3.6.10)], *this defines an adjoint pair $(^*,_*)$ of monoidal Δ-pseudofunctors on* Sp.

Proof. We verify the hypotheses of [LH09, (3.6.10)], following the proof of the [LH09, Scholium, (3.6.10)] where appropriate. Denote by TriCat the 2-category of triangulated categories.

(1) The functor $(-)_*\colon$ Sp $\to$ TriCat is covariant in the sense of [LH09, (3.6.5)]. It is easy to see that $c_{1,g}=c_{f,1}$ are identities. To see that [LH09, (3.6.5.1)] commutes, observe that [LH09, (3.6.3)$_*$] holds for underived pushforward of complexes of modules on the small étale sites. Indeed, the proof of [LH09, (3.6.3)$_*$] depends only on functoriality of composition of direct images of morphisms of ringed topoi. Now, [Sta26, Tags 0D6E & 07A5] can be used to show that the analog of [LH09, (3.6.4.1)] holds in our setting. Hence, [LH09, (3.6.5.1)] commutes.
(2) The functor $(-)^*\colon$ Sp $\to$ TriCat is contravariant in the sense of [LH09, (3.6.5)]. It is easy to see that $d_{1,g}=d_{f,1}$ are identities. We check that the analog of [LH09, (3.6.5.2)] commutes. Note that [Sta26, Tag 0D6D] gives the analog of [LH09, (3.6.4)*]. Since derived pullback preserves K-flat resolutions [Sta26, Tag 0G7E], the analog of [LH09, (3.6.3)*] with derived pullbacks holds, yielding [LH09, (3.6.5.2)].
(3) We have explained [LH09, (3.6.7)(a,b,c) & (3.6.7)$(d(i,ii))$]. However, [LH09, (3.6.7.2)] is [Sta26, Tag 0FPN], [LH09, (3.6.7.1)] is formal because the proof depends only on properties of closed symmetric monoidal categories, and [LH09, Definition 3.4.2] are [Sta26, Tags 0FPK, 0FPL, 0FPM, & 0FPN].
(4) [LH09, (3.6.7)$(d(iii))$] follows by applying [LH09, Lemma-Definition 3.3.5] to see that the derived functor versions of [LH09, (3.6.2) & (3.6.2)^{op}] hold.

(5) [LH09, 3.6.7($d(iv)$)] follows from [Sta26, Tag 0B6C], whereas [LH09, 3.6.7($d(v)$)] is [Sta26, Tag 07A6].

□

Lemma 2.9. *Consider a faithfully flat morphism $f\colon Y \to X$ of algebraic spaces. Then $\mathbf{L}f^*\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ is conservative.*

Proof. Let $E \in D_{\mathrm{qc}}(X)$ satisfy $\mathbf{L}f^*E \cong 0$. Since f is flat, we have $f^*\mathcal{H}^j(E) \cong \mathcal{H}^j(\mathbf{L}f^*E)$ for all $j \in \mathbb{Z}$. Thus, it suffices to check that $f^*\mathcal{H}^j(E) \cong 0$ implies $\mathcal{H}^j(E) \cong 0$ for all $j \in \mathbb{Z}$.

Fix an étale presentation $s\colon U \to X$ from a scheme. Consider the projection morphisms $f'\colon Y \times_X U \to U$ and $s'\colon Y \times_X U \to Y$. By base change, f' is faithfully flat and s' is faithfully flat. Let $t\colon V \to Y \times_X U$ be an étale presentation from a scheme. Then $t^*(s')^*f^*\mathcal{H}^j(E) \cong 0$ for all $j \in \mathbb{Z}$, which is equivalent to $t^*(f')^*s^*\mathcal{H}^j(E) \cong 0$ for all $j \in \mathbb{Z}$. Since $f' \circ t$ is a faithfully flat morphism of schemes, it follows that $s^*\mathcal{H}^j(E) \cong 0$ for all $j \in \mathbb{Z}$ (see e.g. [GW20, Proposition 14.11]). Finally, since s is an étale presentation, $\mathcal{H}^j(E) \cong 0$ for all $j \in \mathbb{Z}$ [Sta26, Tag 07TY]. □

Lemma 2.10. *Let X be a quasi-compact quasi-separated algebraic space. For any $E \in D_{\mathrm{qc}}(X)$, the following are equivalent:*

(1) $E \cong 0$
(2) $\mathbf{L}f^*E \cong 0$ for all flat morphisms $f\colon Y \to X$ of algebraic spaces
(3) $\mathbf{L}f^*E \cong 0$ for some smooth surjective morphism $f\colon Y \to X$ of algebraic spaces
(4) there exists a flat surjective morphism $f\colon Y \to X$ of algebraic spaces such that $\mathbf{L}f^*E \cong 0$.

In particular, if $\mathcal{H}^j(\mathbf{L}f^*E) \cong 0$ for some j, then $\mathcal{H}^j(E) \cong 0$ whenever $f\colon Y \to X$ is faithfully flat.

Proof. It is straightforward to check that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). We show that (4) $\implies$ (1). Assume there exists a flat surjective morphism $f\colon Y \to X$ of algebraic spaces such that $\mathbf{L}f^*E \cong 0$. Let $s\colon U \to X$ be a flat surjective morphism from a scheme. Consider the fibered square

$$\begin{tikzcd}
Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\
Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

Choose an étale presentation $t\colon V \to Y \times_X U$. By base change, f' is flat and surjective. Hence, $f' \circ t$ is a flat surjective morphism of schemes. Since $\mathbf{L}f^*E \cong 0$, it follows that $\mathbf{L}(f \circ s' \circ t)^*E \cong 0$. However, $f \circ s' \circ t = s \circ f' \circ t$, and so $\mathbf{L}(s \circ f' \circ t)^*E \cong 0$. By Lemma 2.9, it follows that $E \cong 0$ because $s \circ f' \circ t$ is faithfully flat. The last claim follows from the fact that $\mathcal{H}^j(\mathbf{L}f^*E) \cong f^*\mathcal{H}^j(E)$ and Lemma 2.9. □

Lemma 2.11. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Choose $E, G \in D(X)$. Assume either E is perfect, or E is pseudocoherent and both $G, \mathbf{L}f^*G$ are bounded below. Then the natural morphism (see [Sta26, Tag 08JF])*

$$\rho_f\colon \mathbf{L}f^*\,\mathbb{R}\mathcal{H}om(E,G) \to \mathbb{R}\mathcal{H}om(\mathbf{L}f^*E, \mathbf{L}f^*G)$$

is an isomorphism. In particular, this morphism exists in $D_{\mathrm{qc}}(Y)$.

Proof. Let $p\colon U \to X$ be an étale presentation. Form the fibered square

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Choose an étale presentation $p''\colon V \to Y \times_X U$. Set $q := p' \circ p''$ and $g := f' \circ p''$. Note that q is an étale presentation. By Lemma 2.8 and [LH09, Exercise 3.7.1.1], there exists a commutative diagram,

$$\begin{tikzcd} \mathbf{L}q^*\mathbf{L}f^*\,\mathbb{R}\mathcal{H}om(E,G) \arrow[r, "\mathbf{L}q^*\rho_f"] \arrow[d, "\cong"'] & \mathbf{L}q^*\,\mathbb{R}\mathcal{H}om(\mathbf{L}f^*E, \mathbf{L}f^*G) \arrow[d, "\rho_q"] \\ \mathbf{L}g^*\mathbf{L}p^*\,\mathbb{R}\mathcal{H}om(E,G) \arrow[d, "\mathbf{L}g^*\rho_p"'] & \mathbb{R}\mathcal{H}om(\mathbf{L}q^*\mathbf{L}f^*E, \mathbf{L}q^*\mathbf{L}f^*G) \arrow[d, "\cong"] \\ \mathbf{L}g^*\,\mathbb{R}\mathcal{H}om(\mathbf{L}p^*E, \mathbf{L}p^*G) \arrow[r, "\rho_g"'] & \mathbb{R}\mathcal{H}om(\mathbf{L}g^*\mathbf{L}p^*E, \mathbf{L}g^*\mathbf{L}p^*G). \end{tikzcd}$$

By [Sta26, Tags 04LX & 08JB], ρ_q and ρ_p are isomorphisms. Since q is faithfully flat, Lemma 2.10 says it suffices to prove that ρ_g is an isomorphism. Applying [Sta26, Tag 08GH], we can reduce to checking the claim on the small Zariski sites of U and V. By [Sta26, Tag 071Q], the conditions on bounded below remain. To see the claim, we can argue exactly as in the case of [GW23, Proposition 22.70], where 'f has finite tor-dimension' can be replaced with our hypothesis that $\mathbf{L}f^*G$ is bounded below. Indeed, this is the only part in the proof of loc. cit. which requires the assumption of finite tor-dimension. This completes the proof of the first claim. The second claim is immediate from [Sta26, Tag 0A8A]. □

2.3.8. *Quasi-affine diagonal.* We record a few straightforward facts.

Lemma 2.12. *Consider a commutative diagram of algebraic spaces*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "h"'] & Y \arrow[d, "f"] \\ & X. \end{tikzcd}$$

If both h and Δ_f are quasi-affine, then g is quasi-affine.

Proof. There exists a fibered square

$$\begin{tikzcd} Z \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "h"'] & X. \end{tikzcd}$$

By base change, p_2 is quasi-affine [Sta26, Tag 03WO]. Moreover, the morphism $(1,g)\colon Z \to Z \times_X Y$ is the base change of the diagonal $\Delta_f\colon Y \to Y \times_X Y$ by the morphism $Z \times_X Y \to Y \times_X Y$. Again, by base change, $(1,g)\colon Z \to Z \times_X Y$ is quasi-affine. Since $g = p_2 \circ (1,g)$, it must be itself quasi-affine [Sta26, Tag 03WN]. □

Lemma 2.13. *Let X be a quasi-separated algebraic space. Then every morphism to X from a quasi-affine scheme is quasi-affine.*

Proof. Note that quasi-separated algebraic spaces have quasi-affine diagonal [Sta26, Tags 03HK, 05W7, & 082J]. Thus, the claim follows from Lemma 2.12. □

3. Relative perfection

3.1. **Characterization of perfectness.** We prove a version of [AJS23, Theorem 2.3] for algebraic spaces. The main result is Proposition 3.12.

Lemma 3.1. *Let X be a quasi-compact quasi-separated algebraic space. Then $\mathbf{R}i_*E \in D^b_{\mathrm{qc}}(X)$ for all $E \in D^b_{\mathrm{qc}}(\operatorname{Spec}(k))$ where $i\colon \operatorname{Spec}(k) \to X$ represents some $p \in |X|$.*

Proof. Since X is quasi-separated it follows that $\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{qc}}(X)$. Indeed, i must be an affine morphism [Sta26, Tag 09TF], so $\mathbf{R}^j i_*\mathcal{O}_{\operatorname{Spec}(k)} \cong 0$ for $j \neq 0$. Now, any $E \in D^b_{\mathrm{qc}}(\operatorname{Spec}(k))$ is a small coproduct of shifts of $\mathcal{O}_{\operatorname{Spec}(k)}$. In particular, $E \cong \oplus_{\alpha\in I}\mathcal{O}^{\oplus\alpha}_{\operatorname{Spec}(k)}[n_\alpha]$ where I is some finite set of cardinals. As $\mathbf{R}i_*\mathcal{O}^{\oplus\alpha}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{qc}}(X)$ for each $\alpha \in I$, it follows that $\mathbf{R}i_*E \in D^b_{\mathrm{qc}}(X)$. □

Lemma 3.2. *Let $f: Y \to X$ be a quasi-affine morphism of quasi-compact quasi-separated algebraic spaces. Then $\mathbf{R}f_*\colon D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X)$ is conservative (i.e. $\mathbf{R}f_*E \cong 0$ implies $E \cong 0$).*

Proof. There exists the canonical factorization

$$Y \xrightarrow{j} \underline{\operatorname{Spec}}_X(f_*\mathcal{O}_Y) \xrightarrow{f'} X.$$

See [Sta26, Tag 081X]. Since f' is affine, $\mathbf{R}f'_*$ is conservative [Sta26, Tag 0E4R]. Moreover, f is quasi-affine, $Y \to \underline{\operatorname{Spec}}_X(f_*\mathcal{O}_Y)$ is an open immersion [Sta26, Tag 086S]. Hence, $\mathbf{R}j_*$ is conservative since $\mathbf{L}j^*\mathbf{R}j_*E \cong E$ for all $E \in D_{\mathrm{qc}}(Y)$. Thus, the claim follows. □

Lemma 3.3. *Let X be a quasi-compact quasi-separated algebraic space. Suppose $i\colon \operatorname{Spec}(k) \to X$ represents some $p \in |X|$. If $E \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{qc}}(X)$, then $\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(\operatorname{Spec}(k))$.*

Proof. The projection formula yields

$$\mathbf{R}i_*\mathbf{L}i^*E \cong E \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}.$$

Hence, by hypothesis, $\mathbf{R}i_*\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(X)$. Moreover, affineness of i implies $\mathcal{H}^j(\mathbf{R}i_*A) \cong i_*\mathcal{H}^j(A)$ for all $j \in \mathbb{Z}$. Thus, Lemma 3.2 gives the boundedness of $\mathbf{L}i^*E$. □

Lemma 3.4. *Let X be a quasi-compact quasi-separated algebraic space. Choose $E \in D^b_{\mathrm{qc}}(X)$ pseudocoherent. If $\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(\operatorname{Spec}(k))$ for all $i\colon \operatorname{Spec}(k) \to X$ that represents any $p \in |X|$, then $\mathbf{R}\mathcal{H}om(E,A) \in D^b_{\mathrm{qc}}(X)$ for all $A \in D^b_{\mathrm{qc}}(X)$.*

Proof. Consider an étale presentation $s\colon U \to X$ from an affine scheme. Observe that for any $q \in U$, with the natural morphism $i\colon \operatorname{Spec}(\kappa(q)) \to U$, the hypothesis says $\mathbf{L}(s \circ i)^*E \in D^b_{\mathrm{qc}}(\operatorname{Spec}(\kappa(q)))$. Hence, by [AJS23, Theorem 2.3], $\mathbb{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{L}s^*A) \in D^b_{\mathrm{qc}}(U)$ for any $A \in D^b_{\mathrm{qc}}(X)$ (e.g. apply [Sta26, Tags 071Q, 08GH, & 08JP]). It follows that $\mathbb{R}\mathcal{H}om(E,A) \in D^b_{\mathrm{qc}}(X)$. Thus, Lemma 2.7 shows that $\mathbf{R}\mathcal{H}om(E,A) \cong \mathbb{R}\mathcal{H}om(E,A)$, which completes the proof. □

Lemma 3.5. *Let X be a quasi-compact quasi-separated algebraic space. Then $E \in D(X)$ is perfect if, and only if, there exists an étale presentation $s \colon U \to X$ such that $\mathbf{L}s^*E$ is perfect in $D(U)$.*

Proof. By [Sta26, Tag 08HG], perfectness of complexes can be detected on the small Zariski site or small étale site of a scheme. Let $E \in D(X)$ be perfect. Choose any étale presentation $s \colon U \to X$. By [Sta26, Tag 08H6], $\mathbf{L}s^*E$ is perfect. We check the converse.

Assume there exists an étale presentation $s \colon U \to X$ such that $\mathbf{L}s^*E$ is perfect in $D(U)$. Choose an étale morphism $t \colon V \to X$. Consider the projections $p \colon V \times_X U \to V$ and $q \colon V \times_X U \to U$ from the fiber product. Since $\mathbf{L}s^*E$ is perfect, it follows that $\mathbf{L}(s \circ q)^*E$ is perfect. Hence, $\mathbf{L}(t \circ p)^*E$ is perfect. Let $r \colon W \to V \times_X U$ be an étale presentation. Then $\mathbf{L}(t \circ p \circ r)^*E$ is perfect. However, $p \circ r$ is an étale covering of V in $X_{\text{étale}}$, and so $\mathbf{L}t^*E$ is perfect [Sta26, Tags 04LX & 08G5]. ◻

Lemma 3.6. *Let X be a quasi-compact quasi-separated algebraic space. Consider $E \in D^b_{\mathrm{qc}}(X)$ pseudocoherent. If $\mathbf{R}\mathcal{H}om(E, \mathbf{R}i_*\mathcal{O}_{\mathrm{Spec}(k)}) \in D^b_{\mathrm{qc}}(X)$ for any $p \in |X|$ and for $i \colon \mathrm{Spec}(k) \to X$ that represents p, then E is perfect.*

Proof. Consider an étale presentation $s \colon U \to X$ from an affine scheme. Fix $q \in U$. Let $i \colon \mathrm{Spec}(k) \to U$ be the natural morphism where $k := \kappa(q)$. There exists a commutative diagram obtained by pullbacks and natural morphisms,

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[dr, "h"] \arrow[drrr, bend left, "i"] \arrow[ddr, bend right, "1_{\mathrm{Spec}(k)}"'] & & & \\
& \mathrm{Spec}(k) \times_X U \arrow[r, "i'"] \arrow[d, "s'"'] & U \times_X U \arrow[r, "q_2"] \arrow[d, "q_1"'] & U \arrow[d, "s"] \\
& \mathrm{Spec}(k) \arrow[r, "i"'] & U \arrow[r, "s"'] & X.
\end{tikzcd}$$

Since X is quasi-separated, $s \circ i$ is affine [Sta26, Tag 09TF]. Hence, $q_2 \circ i'$ is affine by base change, and thus, $\mathrm{Spec}(k) \times_X U$ is an affine scheme. This implies that s' is an affine, and hence, separated morphism. As $1_{\mathrm{Spec}(k)}$ is proper and s' is separated, it follows that h is proper. Consequently, $\mathbf{R}h_*\mathcal{O}_{\mathrm{Spec}(k)} \in D^b_{\mathrm{coh}}(\mathrm{Spec}(k) \times_X U)$. Moreover, s' is smooth because it is the base change of s along $s \circ i$. Since $\mathrm{Spec}(k) \times_X U$ is affine and regular, we have that $\mathbf{R}h_*\mathcal{O}_{\mathrm{Spec}(k)} \in \langle \mathcal{O}_{\mathrm{Spec}(k) \times_X U} \rangle$. Then affineness of $q_2 \circ i'$ yields

$$\mathbf{R}(q_2 \circ i')_*\mathcal{O}_{\mathrm{Spec}(k) \times_X U} \cong (q_2 \circ i')_*\mathcal{O}_{\mathrm{Spec}(k) \times_X U} \in D^b_{\mathrm{qc}}(U).$$

By flat base change,

$$\begin{aligned}
\mathbf{R}(q_2 \circ i')_*\mathcal{O}_{\mathrm{Spec}(k) \times_X U} &\cong \mathbf{R}(q_2 \circ i')_*\mathbf{L}(s')^*\mathcal{O}_{\mathrm{Spec}(k)} \\
&\cong \mathbf{L}s^*\mathbf{R}(s \circ i)_*\mathcal{O}_{\mathrm{Spec}(k)}.
\end{aligned}$$

Since E is pseudocoherent and $\mathbf{R}(s \circ i)_*\mathcal{O}_{\mathrm{Spec}(k)} \in D^+_{\mathrm{qc}}(X)$ (i.e. is bounded below), Lemma 2.11 yields an isomorphism,

$$\mathbf{L}s^*\, \mathbf{R}\mathcal{H}om(E, \mathbf{R}(s \circ i)_*\mathcal{O}_{\mathrm{Spec}(k)}) \to \mathbf{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{L}s^*\mathbf{R}(s \circ i)_*\mathcal{O}_{\mathrm{Spec}(k)}).$$

It follows that

$$\begin{aligned}\mathbf{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}) &\cong \mathbf{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{R}(q_2 \circ i')_*\mathbf{R}h_*\mathcal{O}_{\operatorname{Spec}(k)}) \\ &\in \langle \mathbf{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{R}(q_2 \circ i')_*\mathcal{O}_{\operatorname{Spec}(k)\times_X U}) \rangle \\ &\in \langle \mathbf{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{L}s^*\mathbf{R}(s \circ i)_*\mathcal{O}_{\operatorname{Spec}(k)}) \rangle \\ &\in \langle \mathbf{L}s^* \, \mathbf{R}\mathcal{H}om(E, \mathbf{R}(s \circ i)_*\mathcal{O}_{\operatorname{Spec}(k)}) \rangle.\end{aligned}$$

By hypothesis,

$$\mathbf{R}\mathcal{H}om(E, \mathbf{R}(s \circ i)_*\mathcal{O}_{\operatorname{Spec}(k)}) \in D^b_{\operatorname{qc}}(X),$$

and hence,

$$\mathbf{L}s^* \, \mathbf{R}\mathcal{H}om(E, \mathbf{R}(s \circ i)_*\mathcal{O}_{\operatorname{Spec}(k)}) \in D^b_{\operatorname{qc}}(U).$$

However, $D^b_{\operatorname{qc}}(U)$ is thick in $D_{\operatorname{qc}}(U)$, and so,

$$\mathbf{R}\mathcal{H}om(\mathbf{L}s^*E, \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}) \in D^b_{\operatorname{qc}}(U).$$

As $q \in U$ was arbitrary, [AJS23, Theorem 2.3] shows that $\mathbf{L}s^*E$ is perfect (e.g. apply [Sta26, Tags 071Q, 08GH, 08HG, & 08JP]). Thus, via Lemma 3.5, E is perfect. □

Lemma 3.7. *Let $f\colon Y \to X$ be a morphism of Noetherian algebraic spaces and $E \in D_{\operatorname{qc}}(X)$. If E is pseudocoherent (resp. $\operatorname{Perf}(X)$), then $\mathbf{L}f^*E$ is pseudocoherent (resp. $\mathbf{L}f^*E \in \operatorname{Perf}(Y)$). Additionally, if f is faithfully flat, then the converse holds.*

Proof. The first claim follows from [Sta26, Tags 08H4 & 08H6]. Choose any étale morphism $s\colon U \to X$ from a scheme. Consider the fibered square

$$\begin{array}{ccc} Y \times_X U & \xrightarrow{f'} & U \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

Let $t\colon V \to Y \times_X U$ be an étale presentation. By the first claim, $\mathbf{L}(f \circ s' \circ t)^*E$ is pseudocoherent (resp. perfect). Hence, $\mathbf{L}(f' \circ t)^*\mathbf{L}s^*E$ is pseudocoherent (resp. perfect). By [GW23, Proposition 22.52], $\mathbf{L}s^*E$ is pseudocoherent (resp. perfect). Then [Sta26, Tags 08FU] implies E is pseudocoherent (resp. perfect). □

Lemma 3.8. *Let X be a quasi-compact quasi-separated algebraic space. For any pseudocoherent complex E on X, the following are equivalent:*

(1) $\mathbf{L}i^*E \in D^b_{\operatorname{coh}}(k)$ for some $i\colon \operatorname{Spec}(k) \to X$ that represents some $p \in |X|$
(2) $\mathbf{L}t^*E \in D^b_{\operatorname{coh}}(L)$ for any $t\colon \operatorname{Spec}(L) \to X$ that represents $p \in |X|$.

Proof. It is straightforward to check that (2) $\implies$ (1). We prove the converse. Assume $\mathbf{L}i^*E \in D^b_{\operatorname{coh}}(k)$ for some $i\colon \operatorname{Spec}(k) \to X$ that represents some $p \in |X|$. Let $t\colon \operatorname{Spec}(L) \to X$ be any representative of $p \in |X|$. Then there exist a field ℓ and a commutative square

$$\begin{array}{ccc} \operatorname{Spec}(\ell) & \xrightarrow{i'} & \operatorname{Spec}(L) \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t} \\ \operatorname{Spec}(k) & \xrightarrow[i]{} & X \end{array}$$

because i and t are in the same equivalence class. If $\mathbf{L}i^*E \in D^b_{\mathrm{coh}}(k)$, then

$$\mathbf{L}(t \circ i')^*E = \mathbf{L}(i \circ t')^*E \in D^b_{\mathrm{coh}}(\ell).$$

Since i' is faithfully flat, Lemmas 2.10 and 3.7 implies $\mathbf{L}t^*E \in D^b_{\mathrm{coh}}(L)$. □

Lemma 3.9. *Let X be a quasi-compact quasi-separated algebraic space. Suppose $p \in |X|$ is a closed point. For any $E \in D^b_{\mathrm{qc}}(X)$ pseudocoherent, the following are equivalent:*

(1) $\mathbf{L}i^*E \in D^b_{\mathrm{coh}}(k)$ for some $i\colon \mathrm{Spec}(k) \to X$ that represents p

(2) $\mathbf{L}t^*E \in D^b_{\mathrm{coh}}(L)$ for any $t\colon \mathrm{Spec}(L) \to X$ that represents a $q \in |X|$ which is a generalization of p (i.e. $p \in \overline{\{q\}}$).

Proof. It is clear that (2) $\implies$ (1). We prove the converse. Let $s\colon U \to X$ be an étale presentation. Choose any $q \in |X|$ which is a generalization of p. By [Sta26, Tags 03JX & 03K2], there exist $u, v \in U$ such that $s(u) = p$, $s(v) = q$, and v is a generalization of u. Consider the natural morphisms

$$\mathrm{Spec}(\kappa(u)) \xrightarrow{i_u} \mathrm{Spec}(\mathcal{O}_{U,u}) \xrightarrow{s_u} U.$$

If $\mathbf{L}i^*E \in D^b_{\mathrm{coh}}(k)$, then Lemma 3.8 implies that $\mathbf{L}(s \circ s_u \circ i_u)^*E \in D^b_{\mathrm{coh}}(\kappa(u))$. It follows that $\mathbf{L}(s \circ s_u)^*E \in \mathrm{Perf}(\mathcal{O}_{U,u})$ (see e.g. [GLMP25, Proposition 2.4] and apply [Sta26, Tag 08HG]). Denote by $\sigma\colon \mathrm{Spec}(\mathcal{O}_{U,v}) \to \mathrm{Spec}(\mathcal{O}_{U,u})$ and $i_v\colon \mathrm{Spec}(\kappa(v)) \to \mathrm{Spec}(\mathcal{O}_{U,v})$ the natural morphisms. Then $\mathbf{L}(s \circ s_u \circ \sigma)^*E \in \mathrm{Perf}(\mathcal{O}_{U,v})$ and $\mathbf{L}(s \circ s_u \circ \sigma \circ i_v)^*E \in D^b_{\mathrm{coh}}(\kappa(v))$. Note that $s_u \circ \sigma \circ i_v$ is the natural morphism $\mathrm{Spec}(\kappa(v)) \to U$. Since $(s \circ s_u \circ \sigma \circ i_v)(v) = q$, Lemma 3.8 tells us $\mathbf{L}t^*E \in D^b_{\mathrm{coh}}(L)$ for any $t\colon \mathrm{Spec}(L) \to X$ that represents $q \in |X|$. □

Definition 3.10. Let X be an algebraic space. An object $E \in D_{\mathrm{qc}}(X)$ is said to have **finite flat dimension** if $E \otimes^{\mathbf{L}} B \in D^b_{\mathrm{qc}}(X)$ for all $B \in D^b_{\mathrm{qc}}(X)$.

Remark 3.11. By [Sta26, Tag 03I7], a quasi-separated algebraic space is decent. Moreover, [Sta26, Tag 03K3] shows that a decent algebraic space has a Kolmogorov underlying topological space. Also, from [Sta26, Tag 005E], any nonempty quasi-compact Kolmogorov topological space contains a closed point. Hence, any $p \in |X|$ is the generalization of a closed point $p' \in |X|$.

Proposition 3.12. *Let X be a quasi-compact quasi-separated algebraic space. For any $E \in D^b_{\mathrm{qc}}(X)$ pseudocoherent, the following are equivalent:*

(1) E is perfect

(2) E has finite flat dimension

(3) $E \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\mathrm{Spec}(k)} \in D^b_{\mathrm{qc}}(X)$ for all $i\colon \mathrm{Spec}(k) \to X$ that represents any $p \in |X|$

(4) $\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(\mathrm{Spec}(k))$ for all $i\colon \mathrm{Spec}(k) \to X$ that represents any $p \in |X|$

(5) for any $p \in |X|$ there exists a representative $i\colon \mathrm{Spec}(k) \to X$ such that $\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(\mathrm{Spec}(k))$

(6) $\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(\mathrm{Spec}(k))$ for any $i\colon \mathrm{Spec}(k) \to X$ that represents any closed $p \in |X|$

(7) for any closed $p \in |X|$ there exists a representative $i\colon \mathrm{Spec}(k) \to X$ such that $\mathbf{L}i^*E \in D^b_{\mathrm{qc}}(\mathrm{Spec}(k))$

(8) $\mathbf{R}\mathcal{H}om(E, A) \in D^b_{\mathrm{qc}}(X)$ for all $A \in D^b_{\mathrm{qc}}(X)$

(9) $\mathbf{R}\mathcal{H}om(E, \mathbf{R}i_*\mathcal{O}_{\mathrm{Spec}(k)}) \in D^b_{\mathrm{qc}}(X)$ for all $i\colon \mathrm{Spec}(k) \to X$ that represents any $p \in |X|$.

Proof. If E is perfect, then it has finite flat dimension. Indeed, let $s\colon U \to X$ be an étale presentation from an affine scheme. For any $A \in D^b_{\mathrm{qc}}(X)$, we know that $\mathbf{L}s^*E \otimes^{\mathbf{L}} \mathbf{L}s^*A \in D^b_{\mathrm{qc}}(U)$ because $\mathbf{L}s^*E$ is perfect (e.g. use Lemma 3.5 and [AJS23, Theorem 2.3]). Hence, $\mathbf{L}s^*E \otimes^{\mathbf{L}} \mathbf{L}s^*A \cong \mathbf{L}s^*(E \otimes^{\mathbf{L}} A)$ implies that $E \otimes^{\mathbf{L}} A$ has bounded cohomology, and so $E \otimes^{\mathbf{L}} A \in D^b_{\mathrm{qc}}(X)$. Thus, (1) $\implies$ (2).

By Lemma 3.1, we know that $\mathbf{R}i_*\mathcal{O}_{\mathrm{Spec}(k)} \in D^b_{\mathrm{qc}}(X)$ because X is quasi-separated. Hence, (2) $\implies$ (3). Moreover, Lemma 3.3 tells us (3) $\implies$ (4) since X is quasi-separated. As for (4) $\implies$ (8), this is Lemma 3.4. Clearly, (8) $\implies$ (9), whereas (9) $\implies$ (1) is Lemma 3.6.

Lastly, Lemmas 3.8 and 3.9 imply that (4) $\iff$ (5) $\iff$ (6) $\iff$ (7); see [Sta26, Tag 08H4] and Remark 3.11. $\square$

3.2. **Revisiting schemes**. We clarify that all notions of 'relatively perfect' complexes for morphisms of schemes are equivalent. See Proposition 3.16. This result seems to be missing in the literature, e.g. see comments in [AJS23, §1.4] and below [HCS07, Definition 1.3].

Lemma 3.13. *Let $\mathcal{T}$ be a triangulated category. A bounded t-structure $\tau = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on $\mathcal{T}$ is nondegenerate.*

Proof. Let $E \in \cap_{n\in\mathbb{Z}}\mathcal{T}^{\geq n}$. Assume E is not the zero object. Since τ is bounded, there is an $N \geq 0$ such that $E[N] \in \mathcal{T}^{\leq 0}$ and $E[-N] \in \mathcal{T}^{\geq 0}$. This means $E \in \mathcal{T}^{\leq N} \cap \mathcal{T}^{\geq -N}$. However, $E \in \mathcal{T}^{\geq N+1} = \mathcal{T}^{\geq N}[-1]$, and so, $E \in \mathcal{T}^{\leq N} \cap \mathcal{T}^{\geq N+1}$. Since $\operatorname{Hom}(A,B) = 0$ for all $A \in \mathcal{T}^{\leq N}$ and $B \in \mathcal{T}^{\geq N}[-1]$, we have a contradiction. A similar argument shows $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\leq n} = \operatorname{add}(0)$. $\square$

Lemma 3.14. *Let $\mathcal{T}$ be a triangulated category equipped with a bounded t-structure τ. Denote by H^n_τ the n-th cohomology functor with respect to τ. Then any object E in $\mathcal{T}$ is finitely built by $\oplus_{n\in\mathbb{Z}}H^n_\tau(E)$.*

Proof. Fix $E \in \mathcal{T}$. We prove the claim by induction using the truncation triangles

$$\tau^{\leq k-1}E \to \tau^{\leq k}E \to H^k_\tau(E)[-k] \to (\tau^{\leq k-1}E)[1].$$

By Lemma 3.13, τ is nondegenerate. This implies that the zero objects coincide with those satisfying s-th cohomology vanishes for all $s \in \mathbb{Z}$. Hence, $\langle\{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_0$ consists of only zero objects. Thus, we can impose $E \not\cong 0$. Set $k_1 = \min\{n \in \mathbb{Z} : H^n_\tau(E) \not\cong 0\}$. Then $\tau^{\leq k_1-1}E \cong 0$, and so, $\tau^{\leq k_1}E \cong H^{k_1}_\tau(E)[-k_1]$. Assume by induction that there exists an $N \geq 0$ such that for all $0 \leq s \leq N$ we have $\tau^{\leq k_1+s}E \in \langle\{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_{s+1}$. Consider the truncation triangle

$$\begin{aligned}\tau^{\leq k_1+(N+1)-1}E &\to \tau^{\leq k_1+(N+1)}E \\ &\to H^{k_1+(N+1)}_\tau(E)[-(k_1+N+1)] \to (\tau^{\leq k_1+(N+1)-1}E)[1].\end{aligned}$$

By the inductive hypothesis,

$$\tau^{\leq k_1+(N+1)-1}E \in \langle\{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_{N+1},$$

which implies that $\tau^{\leq k_1+(N+1)}E \in \langle\{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_{N+2}$. Since τ is a bounded t-structure, $H^n_\tau(E) \cong 0$ for $|n| \gg 0$, and hence $E \cong \tau^{\leq k_1+L}E$ for $L \gg 0$. This completes the proof. $\square$

Lemma 3.15. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. For any $E \in D^b_{\mathrm{qc}}(Y)$, the following are equivalent:*

(1) $E \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{qc}}(X) \subseteq D^b_{\mathrm{qc}}(Y)$
(2) *there exists* $[a,b] \subseteq \mathbb{Z}$ *such that* $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M) \cong 0$ *for all* $j \notin [a,b]$ *and* $M \in \mathrm{Qcoh}(X)$.

Proof. First, we show (1) $\implies$ (2). Assume the contrary. Then for each $n \geq 1$ there exists an $M_n \in \mathrm{Qcoh}(X)$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M_n) \ncong 0$ for some $j \notin [-n,n]$. Set $M := \oplus_{n\geq 1} M_n$. By hypothesis, $E \otimes^{\mathbf{L}} \mathbf{L}f^* M \in D^b_{\mathrm{qc}}(Y)$, and hence there exists $[a,b] \subseteq \mathbb{Z}$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M) \cong 0$ for all $j \notin [a,b]$. Choose $n > \max\{|a|,|b|\}$. Since $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M_n)$ is a direct summand of $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M)$ for each $n \geq 1$, $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M_n) \cong 0$ for all $j \notin [a,b]$ and $n \geq 1$. This leads to a contradiction. Indeed, for $n > \max\{|a|,|b|\}$ there exists $j \notin [-n,n]$ with $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M_n) \ncong 0$.

Next, we prove that (2) $\implies$ (1). Let $B \in D^b_{\mathrm{qc}}(X)$. By Lemma 3.14, B is finitely built by its cohomology sheaves $\mathcal{H}^j(B)$. Moreover, the hypothesis implies $E \otimes^{\mathbf{L}} \mathbf{L}f^* \mathcal{H}^i(B) \in D^b_{\mathrm{qc}}(Y)$ for all $i \in \mathbb{Z}$. Since $B \in \langle \{E \otimes^{\mathbf{L}} \mathcal{H}^i(B)\}\rangle$, it follows that

$$E \otimes^{\mathbf{L}} \mathbf{L}f^* B \in \langle \{E \otimes^{\mathbf{L}} \mathbf{L}f^* \mathcal{H}^i(B)\}_{i\in\mathbb{Z}}\rangle \subseteq D^b_{\mathrm{qc}}(Y).$$

This completes the proof. □

Proposition 3.16. *Let $f: Y \to X$ be a morphism of quasi-compact quasi-separated schemes. For any $E \in D^b_{\mathrm{qc}}(Y)$, the following are equivalent:*

(1) $E \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{qc}}(X) \subseteq D^b_{\mathrm{qc}}(Y)$
(2) *E has finite tor-dimension as an object of $D(f^{-1}\mathcal{O}_X)$.*

Proof. It suffices to prove the claim on the small Zariski sites. See [Sta26, Tags 071Q & 08GH]. First, we show (1) $\implies$ (2). By Lemma 3.15, there exists $[a,b] \subseteq \mathbb{Z}$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^* M) \cong 0$ for all $j \notin [a,b]$ and $M \in \mathrm{Qcoh}(X)$. Following [LH09, pg. 78, Example 2.6.7], this condition is equivalent to E having finite flat f-amplitude in $[a,b]$ (see loc. cit. for definition). Moreover, this condition is equivalent to the requirement that for each $p \in Y$, E_p is isomorphic in $D(\mathcal{O}_{X,f(p)})$ to a complex of flat $\mathcal{O}_{X,f(p)}$-modules vanishing in degrees outside of $[a,b]$; see comments above [LH09, Eq. 2.7.6.1]. Applying [Ill71, Proposition 3.3], this stalk local condition is equivalent to E having finite tor-dimension in $[a,b]$ as an object of $D(f^{-1}\mathcal{O}_X)$ (see also [GW20, Proposition 21.169 & Lemma 21.171]). It follows that (1) $\implies$ (2). Conversely, the same references show that (2) in this proof implies (2) of Lemma 3.15. Thus, (2) $\implies$ (1). □

3.3. **Relatively perfect.** We introduce a notion of relatively perfect complexes for morphisms of algebraic spaces and study its first properties.

Definition 3.17. Let $f: Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. We say that $E \in D_{\mathrm{qc}}(Y)$ is **relatively perfect over** X, or f**-perfect**, if $E \otimes^{\mathbf{L}} \mathbf{L}f^* A \in D^b_{\mathrm{qc}}(Y)$ for all $A \in D^b_{\mathrm{qc}}(X)$.

Remark 3.18. Any object $E \in D_{\mathrm{qc}}(Y)$ which is f-perfect belongs to $D^b_{\mathrm{qc}}(Y)$.

Lemma 3.19. *Let $f: Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Then $\mathbf{R}f_* D^b_{\mathrm{qc}}(Y) \subseteq D^b_{\mathrm{qc}}(X)$.*

Proof. By Lemma 3.14, every object of $D^b_{\mathrm{qc}}(Y)$ is finitely built by its cohomology sheaves. Hence, we can reduce to checking the claim for every quasi-coherent $\mathcal{O}_Y$-module. Applying

[Sta26, Tag 08FA], we can find some $N \geq 0$ such that $\mathbf{R}^i f_* M \cong 0$ for all $M \in \mathrm{Qcoh}(Y)$ and $i > N$. This completes the proof. $\square$

Proposition 3.20. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Suppose that $s\colon U \to X$ is a flat affine morphism from a quasi-compact quasi-separated algebraic space. Consider the fibered square*

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

If $E \in D_{\mathrm{qc}}(Y)$ is f-perfect, then $\mathbf{L}(s')^*E$ is f'-perfect.

Proof. Note that $E \in D^b_{\mathrm{qc}}(Y)$. By Lemma 3.19, we have $\mathbf{R}s_* D^b_{\mathrm{qc}}(U) \subseteq D^b_{\mathrm{qc}}(X)$. Since E is f-perfect, it follows that

$$E \otimes^{\mathbf{L}} \mathbf{L}f^* \mathbf{R}s_* D^b_{\mathrm{qc}}(U) \subseteq D^b_{\mathrm{qc}}(Y).$$

From flat base change, we obtain that

$$E \otimes^{\mathbf{L}} \mathbf{R}s'_* \mathbf{L}(f')^* D^b_{\mathrm{qc}}(U) \subseteq D^b_{\mathrm{qc}}(Y).$$

Now, via projection formula, for any $A \in D^b_{\mathrm{qc}}(U)$ there exists an isomorphism

$$E \otimes^{\mathbf{L}} \mathbf{R}s'_* \mathbf{L}(f')^* A \cong \mathbf{R}s'_*(\mathbf{L}(f')^* A \otimes^{\mathbf{L}} \mathbf{L}(s')^* E).$$

Since s' is affine (e.g. use base change), $\mathbf{R}(s')_*$ is conservative (see Lemma 3.2) and t-exact with respect to the standard t-structures (see [Sta26, Tag 073H]). Hence,

$$\mathbf{R}s'_*(\mathbf{L}(f')^* A \otimes^{\mathbf{L}} \mathbf{L}(s')^* E) \in D^b_{\mathrm{qc}}(Y)$$

implies

$$\mathbf{L}(f')^* A \otimes^{\mathbf{L}} \mathbf{L}(s')^* E \in D^b_{\mathrm{qc}}(Y \times_X U).$$

Indeed, $\mathbf{R}^j s'_* B \cong s'_* \mathcal{H}^j(B)$ follows from t-exactness, whereas conservativeness gives bounded cohomology. Therefore, $\mathbf{L}(s')^*E$ is f'-perfect. $\square$

Proposition 3.21. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Suppose that $s\colon U \to X$ is a flat surjective morphism of algebraic spaces. Consider the fibered square*

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

If $\mathbf{L}(s')^*E$ is f'-perfect where $E \in D_{\mathrm{qc}}(Y)$, then E is f-perfect.

Proof. By base change, s' is flat and surjective. As $\mathbf{L}(s')^*E$ is f'-perfect, it follows that $E \in D^b_{\mathrm{qc}}(Y)$. Let $A \in D^b_{\mathrm{qc}}(X)$. Since $\mathbf{L}(s')^*E$ is f'-perfect, it follows that

$$\mathbf{L}(f')^* \mathbf{L}s^* A \otimes^{\mathbf{L}} \mathbf{L}(s')^* E \in D^b_{\mathrm{qc}}(Y \times_X U).$$

This implies

$$\mathbf{L}(f \circ s')^* A \otimes^{\mathbf{L}} \mathbf{L}(s')^* E \in D^b_{\mathrm{qc}}(Y \times_X U).$$

Yet, by Lemma 2.10, $E \otimes^{\mathbf{L}} \mathbf{L}f^*A \in D^b_{\mathrm{qc}}(Y)$. Hence, E is f-perfect. □

Corollary 3.22. *Let $f: Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Choose $E \in D_{\mathrm{qc}}(Y)$. Assume $\mathbf{L}q^*E$ is g-perfect for every commutative diagram*

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "q"'] & U \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where p and q are flat surjective morphisms from quasi-compact quasi-separated schemes. Then E is f-perfect.

Proof. Note that $E \in D^b_{\mathrm{qc}}(Y)$. Let $p: U \to X$ be an étale presentation. Consider the fibered square

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Choose an étale presentation $t: V \to Y \times_X U$. By assumption, we know that $\mathbf{L}(p' \circ t)^*E$ is $(f' \circ t)$-perfect. Hence, for every $B \in D^b_{\mathrm{qc}}(X)$, it follows that

$$\mathbf{L}(p' \circ t)^*E \otimes^{\mathbf{L}} \mathbf{L}(f \circ p' \circ t)^*B \in D^b_{\mathrm{qc}}(V).$$

Since t is flat and surjective, Lemma 2.10 gives

$$\mathbf{L}(p')^*E \otimes^{\mathbf{L}} \mathbf{L}(f \circ p')^*B \in D^b_{\mathrm{qc}}(Y \times_X U).$$

However, by similar reasoning, we obtain that $E \otimes^{\mathbf{L}} \mathbf{L}f^*B \in D^b_{\mathrm{qc}}(Y)$. □

Theorem 3.23. *Let X be a quasi-compact quasi-separated algebraic space with affine diagonal. Consider a quasi-compact quasi-separated morphism $f: Y \to X$ of algebraic spaces. Then $E \in D_{\mathrm{qc}}(Y)$ is f-perfect if, and only if, $\mathbf{L}q^*E$ is g-perfect for every commutative diagram*

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "q"'] & U \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where p and q are étale surjective morphisms from quasi-compact quasi-separated schemes.

Proof. In both cases, E has bounded cohomology, and so we can impose this without loss of generality in the proof. First, let $E \in D^b_{\mathrm{qc}}(Y)$ be f-perfect. Form a commutative diagram

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "q"'] & U \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where p and q are étale surjective morphisms from quasi-compact quasi-separated schemes. If needed, choose $s\colon U' \to U$ an étale presentation from an affine scheme. We show that $\mathbf{L}q^*E$ is g-perfect. Consider the commutative diagram

$$\begin{tikzcd}
Y \times_X U' \arrow[r, "f''"] \arrow[d, "s'"'] & U' \arrow[d, "s"] \\
Y \times_X U \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\
Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

Since X has affine diagonal, $p \circ s$ is affine (see [Sta26, Tag 08GB]). Moreover, as $p \circ s$ is flat, Proposition 3.20 implies that $\mathbf{L}(p' \circ s')^*E$ is f''-perfect. Next, Proposition 3.21 applied to f' and s, shows $\mathbf{L}(p')^*E$ is f'-perfect. Since g factors through f' by a morphism $h\colon V \to Y \times_X U$, it follows that $\mathbf{L}(p' \circ h)^*E$ is $f' \circ h$-perfect because h is étale [Sta26, Tag 05W3], which shows the desired claim. The converse direction follows from Corollary 3.22. $\square$

3.4. **Relatively quasi-perfect.** We introduce a notion of relatively quasi-perfect complexes for morphisms of algebraic spaces. This is motivated by [Bal09, §3]. We study their basic properties. The main result shows that quasi-perfect complexes give rise to useful adjunctions on D_{qc}. See Theorem 3.37.

Definition 3.24. Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. We say that $E \in D_{\mathrm{qc}}(Y)$ is **relatively quasi-perfect over** X, or f**-quasi-perfect**, if $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \in \operatorname{Perf}(X)$ for all $P \in \operatorname{Perf}(Y)$.

Lemma 3.25. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. If E is f-quasi-perfect, then $E \otimes^{\mathbf{L}} Q$ is f-quasi-perfect for all $Q \in \operatorname{Perf}(Y)$. Moreover, if E is pseudocoherent, then $\mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X) \in D^+_{\mathrm{qc}}(Y)$. Lastly, if $f^\times \mathcal{O}_X$ has coherent cohomology and Y is Noetherian, then $\mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X)$ is pseudocoherent (and hence, is in $D^b_{\mathrm{coh}}(Y)$).*

Proof. If E is f-quasi-perfect, then $E \otimes^{\mathbf{L}} Q$ is f-quasi-perfect for all $Q \in \operatorname{Perf}(Y)$, e.g. f-quasi-perfectness is equivalent to

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y)) \subseteq \operatorname{Perf}(X).$$

We check the second claim.

By [Sta26, Tag 0E56], we have $f^\times \mathcal{O}_X \in D^+_{\mathrm{qc}}(Y)$. Moreover, if E is pseudocoherent, it follows that $\mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X) \in D_{\mathrm{qc}}(Y)$ [Sta26, Tag 0A8A]. Furthermore, $\mathbb{R}\mathcal{H}om$ is étale local, meaning it commutes with derived pullback along étale presentations [Sta26, Tags 04LX & 08JB]. Hence, in the case E is pseudocoherent, it follows that $\mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X) \in D^+_{\mathrm{qc}}(Y)$ [Sta26, Tag 0A6H].

Next, we prove that $\mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X) \in D^-_{\mathrm{qc}}(Y)$ if E is pseudocoherent. In other words, $\mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X)$ is bounded and pseudocoherent for any E pseudocoherent and f-quasi-perfect. To check this claim, let G be a compact generator for $D_{\mathrm{qc}}(Y)$ and $n \in \mathbb{Z}$. By adjunction,

$$\begin{aligned}
\operatorname{Hom}(G[n], \mathbb{R}\mathcal{H}om(E, f^\times \mathcal{O}_X)) &\cong \operatorname{Hom}(G[n] \otimes^{\mathbf{L}} E, f^\times \mathcal{O}_X) && \text{(tensor/hom)} \\
&\cong \operatorname{Hom}(\mathbf{R}f_*(G \otimes^{\mathbf{L}} E)[n], \mathcal{O}_X) && \text{(right adj. of push).}
\end{aligned}$$

Since E is f-quasi-perfect, $\mathbf{R}f_*(G \otimes^{\mathbf{L}} E) \in \operatorname{Perf}(X)$. Then [Sta26, Tag 0GFH] asserts that $\operatorname{Hom}(\mathbf{R}f_*(G \otimes^{\mathbf{L}} E)[-N], \mathcal{O}_X) \cong 0$ for $N \gg 0$. A further application of [Sta26, Tag 0GFH] shows $\mathbb{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X) \in D^b_{\mathrm{qc}}(Y)$.

Lastly, we show if $f^{\times}\mathcal{O}_X$ has coherent cohomology and Y is Noetherian, it follows that $\mathbb{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X)$ has coherent cohomology. This follows from [Sta26, Tag 0DoS]. □

Lemma 3.26. *Let S be a quasi-compact quasi-separated algebraic space. Suppose $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ are morphisms of algebraic spaces. Denote by $p_i \colon Y_1 \times_S Y_2 \to Y_i$ the natural projection. For any $E \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$, the following are equivalent:*

(1) E is p_2-quasi-perfect
(2) $\Phi_E(\operatorname{Perf}(Y_1)) \subseteq \operatorname{Perf}(Y_2)$.

Proof. First, assume E is p_2-quasi-perfect. Since $\mathbf{L}p_1^* \operatorname{Perf}(Y_1) \subseteq \operatorname{Perf}(Y_1 \times_S Y_2)$, we have $\Phi_E(\operatorname{Perf}(Y_1)) \subseteq \operatorname{Perf}(Y_2)$. We check the converse.

By [Sta26, Tag 09IY], each $D_{\mathrm{qc}}(Y_i)$ is compactly generated by an object G_i. By Lemma B.1, $\mathbf{L}p_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^* G_2$ is a compact generator for $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Hence, $\mathbf{L}p_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^* G_2$ is a classical generator for $\operatorname{Perf}(Y_1 \times_S Y_2)$ (see [Sta26, Tags 09SR & 09M8]). Therefore,

$$\Phi_E(G_1) \otimes^{\mathbf{L}} G_2 \cong \mathbf{R}(p_2)_*(E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^* G_2) \in \operatorname{Perf}(Y_2).$$

It follows that E is p_2-quasi-perfect. □

Lemma 3.27. *Let $f \colon Y \to X$ be a proper morphism of Noetherian algebraic spaces. Let $E \in D_{\mathrm{qc}}(Y)$ be pseudocoherent. If E is f-perfect, then E is f-quasi-perfect.*

Proof. Throughout, we freely appeal to Lemma 3.19. Since E is f-perfect and pseudocoherent, we know that $E \in D^b_{\mathrm{coh}}(Y)$. Let $P \in \operatorname{Perf}(Y)$. Then $E \otimes^{\mathbf{L}} P \in D^b_{\mathrm{coh}}(Y)$ because $(-) \otimes^{\mathbf{L}} P$ is an endofunctor on $D^b_{\mathrm{coh}}(Y)$. Since f is proper, it follows that $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \in D^b_{\mathrm{coh}}(X)$. Now let $i \colon \operatorname{Spec}(k) \to X$ be a representative of some $p \in |X|$. Then $\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{qc}}(X)$. By projection formula, we have

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} P \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}) \cong \mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}.$$

Since E is f-perfect, we obtain that $E \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{qc}}(Y)$, and so

$$E \otimes^{\mathbf{L}} P \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\mathrm{qc}}(Y).$$

By Lemma 3.19, it follows that

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} P \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}) \in D^b_{\mathrm{qc}}(X).$$

As p was arbitrary, (3) of Proposition 3.12 implies $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \in \operatorname{Perf}(X)$. □

Example 3.28. The two notions of f-perfect and f-quasi-perfectness do not coincide in general. For example, any morphism to a field has structure sheaves being f-perfect. However, if the morphism is not proper, structure sheaves need not be f-quasi-perfect.

Lemma 3.29. *Let $f \colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. For any $E \in D_{\mathrm{qc}}(Y)$ and $P \in D_{\mathrm{qc}}(X)$, there exists an isomorphism*

$$\mathbf{R}f_* \, \mathbf{R}\mathcal{H}om(E, f^{\times}P) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}f_*E, P).$$

Proof. We follow the idea of [ILN15, 1.6.1]. Let $A \in D_{\mathrm{qc}}(X)$. By adjunctions,

$$\begin{aligned}
\mathrm{Hom}(A, \mathbf{R}f_* \mathbf{R}\mathcal{H}om(E, f^\times P)) &\cong \mathrm{Hom}(\mathbf{L}f^*A, \mathbf{R}\mathcal{H}om(E, f^\times P)) && \text{(pull/push)} \\
&\cong \mathrm{Hom}(\mathbf{L}f^*A \otimes^{\mathbf{L}} E, f^\times P) && \text{(tensor/hom)} \\
&\cong \mathrm{Hom}(\mathbf{R}f_*(\mathbf{L}f^*A \otimes^{\mathbf{L}} E), P) && \text{(right adjoint of push)} \\
&\cong \mathrm{Hom}(\mathbf{R}f_*E \otimes^{\mathbf{L}} A, P) && \text{(projection formula)} \\
&\cong \mathrm{Hom}(A, \mathbf{R}\mathcal{H}om(\mathbf{R}f_*E, P)) && \text{(tensor/hom)}.
\end{aligned}$$

This finishes the proof. □

Lemma 3.30. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Consider an f-quasi-perfect pseudocoherent complex E on Y. Then $\mathbf{R}\mathcal{H}om(E, f^\times P)$ is f-quasi-perfect for all $P \in \mathrm{Perf}(X)$.*

Proof. By [Sta26, Tag 0E56], we know that $f^\times P \in D^+_{\mathrm{qc}}(Y)$. Since E is pseudocoherent, it follows that $\mathbb{R}\mathcal{H}om(E, f^\times P) \in D_{\mathrm{qc}}(Y)$ [Sta26, Tag 0A8A]. In this situation, Lemma 2.7 yields

$$\mathbf{R}\mathcal{H}om(E, f^\times P) \cong \mathbb{R}\mathcal{H}om(E, f^\times P).$$

Choose $Q \in \mathrm{Perf}(Y)$. From [Sta26, Tags 08JJ & 0A8A], the double duality morphism $Q \to (Q^\vee)^\vee$ is an isomorphism where $Q^\vee := \mathbf{R}\mathcal{H}om(Q, \mathcal{O}_Y)$, and there exists a natural isomorphism of functors $\mathbf{R}\mathcal{H}om(Q, -) \cong \mathbb{R}\mathcal{H}om(Q, -)$. Then

$$\begin{aligned}
\mathbb{R}\mathcal{H}om(E, f^\times P) \otimes^{\mathbf{L}} Q &\cong \mathbb{R}\mathcal{H}om(E, f^\times P) \otimes^{\mathbf{L}} (Q^\vee)^\vee \\
&\cong \mathbb{R}\mathcal{H}om(Q^\vee, \mathbb{R}\mathcal{H}om(E, f^\times P)) && \text{([Sta26, Tag 08JJ])} \\
&\cong \mathbb{R}\mathcal{H}om(Q^\vee \otimes^{\mathbf{L}} E, f^\times P) && \text{([Sta26, Tag 08J9])}.
\end{aligned}$$

Applying Lemma 2.7,

$$\mathbb{R}\mathcal{H}om(Q^\vee \otimes^{\mathbf{L}} E, f^\times P) \cong \mathbf{R}\mathcal{H}om(Q^\vee \otimes^{\mathbf{L}} E, f^\times P).$$

Moreover, Lemma 3.29 yields an isomorphism

$$\mathbf{R}f_* \mathbf{R}\mathcal{H}om(Q^\vee \otimes^{\mathbf{L}} E, f^\times P) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}f_*(Q^\vee \otimes^{\mathbf{L}} E), P).$$

Since $Q^\vee$ is perfect, the hypothesis implies that $\mathbf{R}f_*(Q^\vee \otimes^{\mathbf{L}} E)$ is perfect. As P is perfect, we conclude that $\mathbf{R}\mathcal{H}om(\mathbf{R}f_*(Q^\vee \otimes^{\mathbf{L}} E), P)$ is perfect, which completes the proof. □

Lemma 3.31. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. For any $E \in D_{\mathrm{qc}}(Y)$, E is f-quasi-perfect if, and only if, $\mathbf{R}\mathcal{H}om(E, f^\times(-))\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ preserves small coproducts.*

Proof. There exists an adjunction

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} (-))\colon D_{\mathrm{qc}}(Y) \leftrightarrows D_{\mathrm{qc}}(X) \colon \mathbf{R}\mathcal{H}om(E, f^\times(-)).$$

Hence, the desired claim follows from [Nee96, Theorem 5.1]. □

Proposition 3.32. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. If $E \in D_{\mathrm{qc}}(Y)$ is f-quasi-perfect, then there exists an isomorphism for all $G, B \in D_{\mathrm{qc}}(X)$,*

$$\mathbf{R}\mathcal{H}om(E, f^\times G) \otimes^{\mathbf{L}} \mathbf{L}f^*B \to \mathbf{R}\mathcal{H}om(E, f^\times(G \otimes^{\mathbf{L}} B)).$$

Proof. We follow the strategy of [Bal09, Lemma 3.5] and add details for convenience. Recall that there exists the adjunction

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} (-)) \colon D_{\mathrm{qc}}(Y) \leftrightarrows D_{\mathrm{qc}}(X) \colon \mathbf{R}\mathcal{H}om(E, f^{\times}(-)).$$

Consider the counit

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G))) \to G.$$

Tensoring with B gives us a morphism

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G))) \otimes^{\mathbf{L}} B \to G \otimes^{\mathbf{L}} B. \tag{3.1}$$

By projection formula, there exists a natural isomorphism

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G))) \otimes^{\mathbf{L}} B \to \mathbf{R}f_*(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G)) \otimes^{\mathbf{L}} \mathbf{L}f^*B). \tag{3.2}$$

Composing the inverse of (3.2) with (3.1) yields

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G)) \otimes^{\mathbf{L}} \mathbf{L}f^*B) \to G \otimes^{\mathbf{L}} B$$

By adjunctions, this corresponds to a morphism

$$\begin{aligned} &\mathrm{Hom}(\mathbf{R}f_*(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G)) \otimes^{\mathbf{L}} \mathbf{L}f^*B), G \otimes^{\mathbf{L}} B) \\ &\cong \mathrm{Hom}(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}(G)) \otimes^{\mathbf{L}} \mathbf{L}f^*B, f^{\times}(G \otimes^{\mathbf{L}} B)) && \text{(right adj. of push)} \\ &\cong \mathrm{Hom}(\mathbf{R}\mathcal{H}om(E, f^{\times}(G)) \otimes^{\mathbf{L}} \mathbf{L}f^*B, \mathbf{R}\mathcal{H}om(E, f^{\times}(G \otimes^{\mathbf{L}} B))) && \text{(tensor/hom)}. \end{aligned} \tag{3.3}$$

This yields the desired morphism. There exists a sequence of natural isomorphisms for each $G \in D_{\mathrm{qc}}(X)$ and $B \in \mathrm{Perf}(X)$:

$$\begin{aligned} &\mathrm{Hom}(A, \mathbf{R}\mathcal{H}om(E, f^{\times}G) \otimes \mathbf{L}f^*B) \\ &\qquad \cong \mathrm{Hom}(A, \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathbf{L}f^*B, \mathcal{O}_Y), \mathbf{R}\mathcal{H}om(E, f^{\times}G))) && \text{([Sta26, Tag 08JJ])} \\ &\qquad \cong \mathrm{Hom}(A \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}f^*B, \mathcal{O}_Y), \mathbf{R}\mathcal{H}om(E, f^{\times}G)) && \text{(tensor/hom)} \\ &\qquad \cong \mathrm{Hom}(A \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}f^*B, \mathcal{O}_Y) \otimes^{\mathbf{L}} E, f^{\times}G) && \text{(tensor/hom)} \\ &\qquad \cong \mathrm{Hom}(\mathbf{R}f_*(A \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}f^*B, \mathcal{O}_Y) \otimes^{\mathbf{L}} E), G) && \text{(right adj. of push)} \\ &\qquad \cong \mathrm{Hom}(\mathbf{R}f_*(A \otimes^{\mathbf{L}} \mathbf{L}f^*(\mathbf{R}\mathcal{H}om(B, \mathcal{O}_X)) \otimes^{\mathbf{L}} E), G) && \text{(Lemma 2.11)} \\ &\qquad \cong \mathrm{Hom}(\mathbf{R}f_*(A \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(B, \mathcal{O}_X), G) && \text{(projection formula)} \\ &\qquad \cong \mathrm{Hom}(\mathbf{R}f_*(A \otimes^{\mathbf{L}} E), \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(B, \mathcal{O}_X), G)) && \text{(tensor/hom)} \\ &\qquad \cong \mathrm{Hom}(\mathbf{R}f_*(A \otimes^{\mathbf{L}} E), B \otimes^{\mathbf{L}} G) && \text{([Sta26, Tag 08JJ])} \\ &\qquad \cong \mathrm{Hom}(A \otimes^{\mathbf{L}} E, f^{\times}(B \otimes^{\mathbf{L}} G)) && \text{(right adj. of push)} \\ &\qquad \cong \mathrm{Hom}(A, \mathbf{R}\mathcal{H}om(E, f^{\times}(B \otimes^{\mathbf{L}} G))) && \text{(tensor/hom)}. \end{aligned}$$

Following the argument of [Bal09, Lemma 3.5], one checks that the end result of this sequence coincides with the precomposition with (3.3). In fact, if we fix B, it follows that

$$\mathbf{R}f_*((-) \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(B, \mathcal{O}_X) \colon D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X)$$

admits a right adjoint. However, the computation above shows there are two functors which are right adjoint. By uniqueness of right adjoints, this gives us a natural isomorphism

$$\mathbf{R}\mathcal{H}om(E, f^{\times}(-)) \otimes \mathbf{L}f^*B \implies \mathbf{R}\mathcal{H}om(E, f^{\times}(B \otimes^{\mathbf{L}} (-)))$$

of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$. Hence, the morphism in (3.3) is an isomorphism if B is perfect. Using Lemma 3.31 and that $D_{\mathrm{qc}}(Y)$ and $D_{\mathrm{qc}}(X)$ are compactly generated, we have that

$$\mathbf{R}\mathcal{H}om(E, f^{\times}(-)) \otimes \mathbf{L}f^*(\#),$$
$$\mathbf{R}\mathcal{H}om(E, f^{\times}((\#) \otimes^{\mathbf{L}} (-)))$$

both commute with small coproducts because $\otimes^{\mathbf{L}}$ and $\mathbf{L}f^*$ do themselves. Consider the smallest triangulated subcategory $\mathcal{T}$ such that an isomorphism occurs in (3.3) if we fix G. It can be checked this is triangulated and closed under small coproducts. Since the compacts are contained in $\mathcal{T}$ and $D_{\mathrm{qc}}(X)$ is compactly generated, it follows that $\mathcal{T} = D_{\mathrm{qc}}(X)$. □

Lemma 3.33. *Let $f \colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Then $E \in D_{\mathrm{qc}}(Y)$ is the zero object if, and only if, $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \cong 0$ for all $P \in \mathrm{Perf}(Y)$.*

Proof. If $E \in D_{\mathrm{qc}}(Y)$ is the zero object, then $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \cong 0$ for all $P \in \mathrm{Perf}(Y)$. We prove the converse. By adjunction,

$$\begin{aligned}
0 &\cong \mathrm{Hom}(\mathcal{O}_X, \mathbf{R}f_*(E \otimes^{\mathbf{L}} P)) \\
&\cong \mathrm{Hom}(\mathbf{L}f^*\mathcal{O}_X, E \otimes^{\mathbf{L}} P) && \text{(pull/push)} \\
&\cong \mathrm{Hom}(\mathcal{O}_Y, \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(P, \mathcal{O}_Y), E)) && \text{([Sta26, Tag 08JJ])} \\
&\cong \mathrm{Hom}(\mathbf{R}\mathcal{H}om(P, \mathcal{O}_Y), E) && \text{([Sta26, Tag 08J7]).}
\end{aligned}$$

It follows from the assumption that $E \cong 0$. Indeed, use that perfect complexes are dualizable in $D_{\mathrm{qc}}(Y)$ [Sta26, Tags 0FPU, 0FPV, 09M8]. In particular, we have shown that $\mathrm{Hom}(Q, E[n]) \cong 0$ for any compact generator Q of $D_{\mathrm{qc}}(Y)$ and $n \in \mathbb{Z}$. □

Lemma 3.34. *Let $f \colon Y \to X$ be a separated finitely presented morphism of Noetherian algebraic spaces. Then $f^! D^+_{\mathrm{coh}}(X) \subseteq D^+_{\mathrm{coh}}(Y)$.*

Proof. Let $E \in D^+_{\mathrm{coh}}(X)$. Choose an étale presentation $s \colon U \to X$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd}
Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\
Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

By Lemmas 2.12 and 2.13, s is separated. Base change implies each morphism in the diagram is finitely presented, and each algebraic space is Noetherian. By Lemma C.3, we have $\mathbf{L}(s')^* f^! E \cong (f')^! \mathbf{L}s^* E$. Observe that $\mathbf{L}s^*E \in D^+_{\mathrm{qc}}(U)$. Choose an étale surjective morphism $t \colon V \to Y \times_X U$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd}
V' \arrow[r, "t_2"] \arrow[d, "t_1"'] & V \arrow[d, "t"] \\
V \arrow[r, "t"'] & Y \times_X U.
\end{tikzcd}$$

By Lemma 2.13, t is quasi-affine. In particular, t is representable by schemes. As V is affine, each t_i is separated and étale with scheme source.

Applying Lemma C.3, there exists an isomorphism

$$t_2^! \mathbf{L}t^* (f')^! \mathbf{L}s^* E \cong \mathbf{L}t_1^* t^! (f')^! \mathbf{L}s^* E.$$

Since t_i is étale, [Sta26, Tag 0FWI] shows that $t_i^! \cong \mathbf{L}t_i^*$. Combining gives the identifications,

$$\mathbf{L}t_2^*\mathbf{L}t^*(f')^!\mathbf{L}s^*E \cong t_2^!\mathbf{L}t^*(f')^!\mathbf{L}s^*E \cong \mathbf{L}t_1^*t^!(f')^!\mathbf{L}s^*E.$$

Note that Lemma C.1 says $(f' \circ t)^!\mathbf{L}s^*E \cong t^!(f')^!\mathbf{L}s^*E$. By [Sta26, Tag 0AU1], it follows that $(f' \circ t)^!\mathbf{L}s^*E \in D^+_{\mathrm{coh}}(V)$, and hence,

$$\mathbf{L}t_2^*\mathbf{L}t^*(f')^!\mathbf{L}s^*E \cong \mathbf{L}t_1^*t^!(f')^!\mathbf{L}s^*E \in D^+_{\mathrm{coh}}(V').$$

Since $t \circ t_2$ is an étale presentation, the derived pullback is t-exact with respect to the standard t-structures, and so Lemma 2.9 shows that $(f')^!\mathbf{L}s^*E \in D^+_{\mathrm{qc}}(Y \times_X U)$. Similar reasoning shows $f^!E \in D^+_{\mathrm{qc}}(Y)$ as desired. Lastly, we check the claim regarding coherent cohomology. However, by [Sta26, Tag 07UB], this follows from the same reasoning. Indeed, it suffices to check that the cohomology sheaf is coherent after pullback along an étale presentation, and so we can use the t-exactness to detect this. □

Proposition 3.35. *Let $f\colon Y \to X$ be a proper morphism of Noetherian algebraic spaces. If $E \in D_{\mathrm{qc}}(Y)$ is f-quasi-perfect and pseudocoherent, then the natural morphism*

$$\nu\colon E \to \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X)$$

is an isomorphism.

Proof. We follow the argument of [Bal09, Lemma 3.9] and add details for convenience. Recall the natural morphism is the one corresponding to the identity via adjunction:

$$\begin{aligned}
&\mathrm{Hom}(E, \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X)) \\
&\cong \mathrm{Hom}(E \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X) && \text{(tensor/hom)} \\
&\cong \mathrm{Hom}(\mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X) \otimes^{\mathbf{L}} E, f^\times\mathcal{O}_X) && \text{(switch)} \\
&\cong \mathrm{Hom}(\mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), \mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X)) && \text{(tensor/hom).}
\end{aligned}$$

Next, for each $P \in \mathrm{Perf}(Y)$, one checks that the diagram commutes

$$\begin{tikzcd}
E \otimes^{\mathbf{L}} P \arrow[r, "\nu_E \otimes^{\mathbf{L}} P"] \arrow[d, "1_{E \otimes^{\mathbf{L}} P}"'] & \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X) \otimes^{\mathbf{L}} P \arrow[d] \\
E \otimes^{\mathbf{L}} P \arrow[r, "\nu_{E \otimes^{\mathbf{L}} P}"'] & \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E \otimes^{\mathbf{L}} P, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X).
\end{tikzcd}$$

The unmarked horizontal morphism arises via Lemma 2.6, which is an isomorphism because P is perfect. To be precise, Lemma C.2 says properness of f implies $f^\times \cong f^!$. By Lemma 3.34, $f^\times\mathcal{O}_X \in D^+_{\mathrm{coh}}(Y)$. Hence, Lemma 3.30 shows that $\mathbb{R}\mathcal{H}om(E, f^\times\mathcal{O}_X)$ is f-quasi-perfect. Then Lemma 3.25 implies

$$\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X) \text{ and } \mathbb{R}\mathcal{H}om(E, f^\times\mathcal{O}_X)$$

have bounded and coherent cohomology. Hence, by Lemma 2.7,

$$\mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X) \cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X)$$

In fact, by similar reasoning with Lemma 3.25,

$$\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E \otimes^{\mathbf{L}} P, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X) \cong \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E \otimes^{\mathbf{L}} P, f^\times\mathcal{O}_X), f^\times\mathcal{O}_X)$$

because $E \otimes^{\mathbf{L}} P$ is pseudocoherent and f-quasi-perfect. Now, Lemma 2.6 and [Sta26, Tag 08J9], give us

$$\begin{aligned}&\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X), f^{\times}\mathcal{O}_X) \otimes^{\mathbf{L}} P \\ &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(P, \mathbb{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X)), f^{\times}\mathcal{O}_X) \\ &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E \otimes^{\mathbf{L}} P, f^{\times}\mathcal{O}_X), f^{\times}\mathcal{O}_X).\end{aligned}$$

Back to the proof. By Lemma 3.33, it suffices to check that the cone of $\mathbf{R}f_*\nu_{E\otimes^{\mathbf{L}}Q}$ is zero for all Q perfect on Y, or equivalently that $\mathbf{R}f_*\nu_{E\otimes^{\mathbf{L}}Q}$ is an isomorphism for all Q perfect on Y. From Lemma 3.29, the natural morphism

$$\eta_A \colon \mathbf{R}f_*\,\mathbf{R}\mathcal{H}om(A, f^!\mathcal{O}_X) \to \mathbf{R}\mathcal{H}om(\mathbf{R}f_*A, \mathcal{O}_X)$$

is an isomorphism. Note that $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_X) \colon D_{\mathrm{qc}}(X)^{op} \to D_{\mathrm{qc}}(X)$ is a functor. Hence, we have a commutative diagram

$$\begin{array}{c} \mathbf{R}f_*(E \otimes^{\mathbf{L}} Q) \\ \Big\downarrow{\scriptstyle \mathbf{R}f_*(\nu_{E\otimes^{\mathbf{L}}Q})} \\ \mathbf{R}f_*\Big(\mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E \otimes^{\mathbf{L}} Q, f^{\times}\mathcal{O}_X), f^{\times}\mathcal{O}_X)\Big) \\ \Big\downarrow{\scriptstyle \eta_{\mathbf{R}\mathcal{H}om(E\otimes^{\mathbf{L}}Q, f^{\times}\mathcal{O}_X)}} \\ \mathbf{R}\mathcal{H}om(\mathbf{R}f_*\,\mathbf{R}\mathcal{H}om(E \otimes^{\mathbf{L}} Q, f^{\times}\mathcal{O}_X), \mathcal{O}_X) \\ \Big\downarrow{\scriptstyle \mathbf{R}\mathcal{H}om(\eta^{-1}_{E\otimes^{\mathbf{L}}Q}, \mathcal{O}_X)} \\ \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathbf{R}f_*(E \otimes^{\mathbf{L}} Q), \mathcal{O}_X), \mathcal{O}_X). \end{array}$$

It can be checked that the morphism above is the double dualization morphism,

$$\mathbf{R}f_*(E \otimes^{\mathbf{L}} Q) \to \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathbf{R}f_*(E \otimes^{\mathbf{L}} Q), \mathcal{O}_X), \mathcal{O}_X).$$

Since E is f-quasi-perfect, $\mathbf{R}f_*(E \otimes^{\mathbf{L}} Q)$ is perfect. Thus, the morphism above must be an isomorphism [Sta26, Tags 08JJ & 0A8A]. □

Remark 3.36. A variation of Proposition 3.35 is proved later in Lemma 4.15 for $f^!$.

Theorem 3.37. *Let $f \colon Y \to X$ be a proper morphism of Noetherian algebraic spaces. If $E \in D_{\mathrm{qc}}(Y)$ is f-quasi-perfect and pseudocoherent, then there exists an adjunction*

$$\mathbf{R}f_*((-) \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X)) \colon D_{\mathrm{qc}}(Y) \leftrightarrows D_{\mathrm{qc}}(X) \colon E \otimes^{\mathbf{L}} \mathbf{L}f^*(-).$$

Proof. This follows from natural isomorphisms for all $A \in D_{\mathrm{qc}}(Y)$ and $G \in D_{\mathrm{qc}}(X)$:

$$\begin{aligned}&\mathrm{Hom}(A, E \otimes^{\mathbf{L}} \mathbf{L}f^*G) \\ &\quad\cong \mathrm{Hom}(A, \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X), f^{\times}\mathcal{O}_X) \otimes^{\mathbf{L}} \mathbf{L}f^*G) && \text{(Proposition 3.35)} \\ &\quad\cong \mathrm{Hom}(A, \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X), f^{\times}G)) && \text{(Proposition 3.32)} \\ &\quad\cong \mathrm{Hom}(A \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X), f^{\times}G) && \text{(tensor/hom)} \\ &\quad\cong \mathrm{Hom}(\mathbf{R}f_*(A \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E, f^{\times}\mathcal{O}_X)), G) && \text{(right adj. of push).}\end{aligned}$$

□

3.5. **Properness.** We show that relative perfectness coincides with relative quasi-perfectness.

Lemma 3.38 (cf. [Sta26, Tags 0GFH & 0GFI]). *Let X be a quasi-compact quasi-separated algebraic space. Consider a closed subset $Z \subseteq |X|$ with quasi-compact complement. Fix a compact generator G for $D_{\mathrm{qc},Z}(X)$. For any $E \in D_{\mathrm{qc},Z}(X)$, the following are equivalent:*

(1) $E \in D^+_{\mathrm{qc},Z}(X)$
(2) there exists $i \geq 0$ such that $\mathrm{Hom}(G[t], E) \cong 0$ if $t \geq i$.

A similar argument holds for objects in $D^-_{\mathrm{qc}}(X)$ where we take $[-t]$.

Proof. We show the case for $Z = |X|$ because the general case can be argued the same. We prove that (1) $\iff$ (2). This may be argued directly from [Sta26, Tags 0GFH & 0GFI] but we provide an alternate proof.

Suppose $E \in D^+_{\mathrm{qc}}(X)$. Then there exists an $n \geq 0$ such that $E \in D^{\geq -n}_{\mathrm{qc}}(X)$. Hence, $E[-n] \in D^{\geq 0}_{\mathrm{qc}}(X) = D^{\geq -n}_{\mathrm{qc}}(X)[-n]$. It follows that $\mathrm{Hom}(A, E[-n-1]) = 0$ for all $A \in D^{\leq 0}_{\mathrm{qc}}(X)$ because $D^{\geq 0}_{\mathrm{qc}}(X)[-1] = (D^{\leq 0}_{\mathrm{qc}}(X))^{\perp}$. Since $D^{\leq 0}_{\mathrm{qc}}(X)$ lies in the preferred equivalence class, there exists an $N \geq 0$ such that $\overline{\langle G \rangle}^{(-\infty,0]}[N] \subseteq D^{\leq 0}_{\mathrm{qc}}(X)$. Thus, $\mathrm{Hom}(G[N], E[-n-1]) = 0$, and hence, $\mathrm{Hom}(G[N+a], E[-n-1]) = 0$ for all $a \geq 0$ because $D^{\leq 0}_{\mathrm{qc}}(X)$ is closed under positive shifts. As $n \geq 0$, it follows $\mathrm{Hom}(G[N+a+n+1], E) = 0$ for all $a \geq 0$. In other words, $\mathrm{Hom}(G[i], E) \cong 0$ for all $i \gg 0$, which shows that (1) $\implies$ (2).

Conversely, suppose that $\mathrm{Hom}(G[i], E) \cong 0$ for all $i \gg 0$. Then there exists a smallest $n \geq 0$ such that this vanishing occurs. Set n to be this integer. By Lemma A.8, there exists $N \geq 0$ satisfying $D^{\leq -N}_{\mathrm{qc}}(X) = D^{\leq 0}_{\mathrm{qc}}(X)[N] \subseteq \overline{\langle G[n] \rangle}^{(-\infty,0]}$. Hence, for all $A \in D^{\leq 0}_{\mathrm{qc}}(X)[N]$, we have $\mathrm{Hom}(A, E) = 0$. It follows that $E \in (D^{\leq 0}_{\mathrm{qc}}(X)[N])^{\perp} = D^{\geq -N}_{\mathrm{qc}}(X)[-1]$. Since $N \geq 0$, we obtain that $D^{\leq -N-a}_{\mathrm{qc}}(X) \subseteq D^{\leq -N}_{\mathrm{qc}}(X)$ for all $a \geq 0$. If necessary, choose $N > 1$. This implies that $E \in D^{\geq -N}_{\mathrm{qc}}(X)$ for some $N \gg 1$. Consequently, $E \in D^+_{\mathrm{qc}}(X)$.

The last claim is argued dually to the argument above. □

Proposition 3.39. *Let $f \colon Y \to X$ be a proper morphism of Noetherian algebraic spaces. Choose $E \in D^-_{\mathrm{coh}}(Y)$. Then E is f-perfect if, and only if, it is f-quasi-perfect. In such a case, E has bounded cohomology.*

Proof. The last claim follows from the fact E is pseudocoherent and f-perfect. Moreover, Lemma 3.27 shows f-perfectness implies f-quasi-perfectness. We check the converse. Choose an étale presentation $s \colon U \to X$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Fix $A \in D^b_{\mathrm{qc}}(X)$. To show $E \otimes^{\mathbf{L}} \mathbf{L}f^*A \in D^b_{\mathrm{qc}}(Y)$, it suffices to check that

$$\mathbf{L}(s')^*(E \otimes^{\mathbf{L}} \mathbf{L}f^*A) \cong \mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(s')^*\mathbf{L}f^*A \in D^b_{\mathrm{qc}}(Y \times_X U)$$

By Lemma 2.12, s is quasi-affine. Hence, s' must be quasi-affine. Then Lemma B.3 shows $\mathbf{L}(s')^* \operatorname{Perf}(Y)$ compactly generates $D_{\mathrm{qc}}(Y \times_X U)$. Applying flat base change, it follows that

$$\begin{aligned}\mathbf{R}f'_*(\mathbf{L}(s')^*E \otimes^{\mathbf{L}} \operatorname{Perf}(Y \times_X U)) &\subseteq \mathbf{R}f'_*(\mathbf{L}(s')^*E \otimes^{\mathbf{L}} \langle \mathbf{L}(s')^* \operatorname{Perf}(Y)\rangle) \\ &\subseteq \langle \mathbf{R}f'_*(\mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(s')^* \operatorname{Perf}(Y))\rangle \\ &\subseteq \langle \mathbf{R}f'_*\mathbf{L}(s')^*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y))\rangle \\ &\subseteq \langle \mathbf{L}s^*\mathbf{R}f_*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y))\rangle.\end{aligned}$$

As E is f-quasi-perfect, $\mathbf{R}f_*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y)) \subseteq \operatorname{Perf}(X)$. Hence, we obtain

$$\langle \mathbf{L}s^*\mathbf{R}f_*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y))\rangle \subseteq \operatorname{Perf}(U),$$

and so $\mathbf{L}(s')^*E$ is f'-quasi-perfect.

Let G be a compact generator for $D_{\mathrm{qc}}(Y \times_X U)$. By Theorem 3.37, we know that

$$\begin{aligned}&\operatorname{Hom}(G, \mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(f')^*\mathbf{L}s^*A) \\ &\quad\cong \operatorname{Hom}(\mathbf{R}f'_*(G \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}(s')^*E, (f')^\times \mathcal{O}_U)), \mathbf{L}(s^*A)).\end{aligned}$$

However, Lemma 3.30 says

$$\mathbf{R}f'_*(G \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}(s')^*E, (f')^\times \mathcal{O}_U)) \subseteq \operatorname{Perf}(U).$$

Since U is affine, $D_{\mathrm{qc}}(U)$ is compactly generated by

$$\mathbf{R}f'_*(G \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}(s')^*E, (f')^\times \mathcal{O}_U)) \oplus \mathcal{O}_U.$$

By Lemma C.2, $h^! \cong h^\times$ for all proper morphisms h of Noetherian algebraic spaces. We apply this below. Now, $\mathbf{L}s^*A \in D^b_{\mathrm{qc}}(U)$, and so Lemma 3.38 implies

$$\operatorname{Hom}((\mathbf{R}f'_*(G \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}(s')^*E, (f')^! \mathcal{O}_U)) \oplus \mathcal{O}_U)[i], \mathbf{L}(s^*A)) \cong 0$$

for all $i \gg 0$. Hence,

$$\operatorname{Hom}(\mathbf{R}f'_*(G \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(\mathbf{L}(s')^*E, (f')^! \mathcal{O}_U))[i], \mathbf{L}(s^*A)) \cong 0.$$

By adjunction, we have

$$\operatorname{Hom}(G[i], \mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(f')^*\mathbf{L}s^*A) \cong 0$$

for all $i \gg 0$. Then Lemma 3.38 shows that $\mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(f')^*\mathbf{L}s^*A \in D^+_{\mathrm{qc}}(Y \times_X U)$. A similar argument shows $\mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(f')^*\mathbf{L}s^*A \in D^-_{\mathrm{qc}}(Y \times_X U)$. Therefore, $\mathbf{L}(s')^*E \otimes^{\mathbf{L}} \mathbf{L}(f')^*\mathbf{L}s^*A \in D^b_{\mathrm{qc}}(Y \times_X U)$, which completes the proof. □

Lemma 3.40. *Let X be a Noetherian scheme. Then $E \in D^b_{\mathrm{coh}}(X)$ is perfect if, and only if, $E \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(\kappa(p))} \in D^b_{\mathrm{coh}}(X)$ for all morphisms $i \colon \operatorname{Spec}(\kappa(p)) \to X$ associated to a closed point $p \in X$.*

Proof. It suffices to prove the claim on the small Zariski site [Sta26, Tags 071Q]. By Proposition 3.12, if E is perfect, then $E \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(\kappa(p))} \in D^b_{\mathrm{coh}}(X)$ for all morphisms $i \colon \operatorname{Spec}(\kappa(p)) \to X$ associated to a closed point $p \in X$. We check the converse. Choose a closed point $p \in X$. Denote by $i \colon \operatorname{Spec}(\kappa(p)) \to X$ the associated closed immersion. The morphism i factors via natural morphisms,

$$\operatorname{Spec}(\kappa(p)) \xrightarrow{i'} \operatorname{Spec}(\mathcal{O}_{X,p}) \xrightarrow{s} X.$$

It follows that

$$\mathbf{L}s^*(E \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(\kappa(p))}) \cong E_p \otimes^{\mathbf{L}} \mathbf{R}i'_*\mathcal{O}_{\operatorname{Spec}(\kappa(p))} \in D^b_{\operatorname{coh}}(\mathcal{O}_{X,p}).$$

Hence, E_p is perfect. Since $p \in X$ was an arbitrary closed point, this implies E is perfect (see e.g. [GLMP25, Proposition 2.4]). □

Remark 3.41. Lemma 3.40 implicitly appears in the proof of [DLM25, Lemma 3.4] when loc. cit. references [AJS23, Theorem 2.3(3)].

Lemma 3.42. *Let X be a Noetherian algebraic space. For any $E \in D^b_{\operatorname{coh}}(X)$, the collection of $p \in |X|$ such that $\mathbf{L}s^*E$ is perfect, where $s\colon \operatorname{Spec}(k) \to X$ is a representative of p, is open in $|X|$.*

Proof. Choose an étale presentation $\pi\colon U \to X$ from an affine scheme. Let S be the collection of $p \in |X|$ such that $\mathbf{L}j_p^*E$ is perfect where $j_p\colon \operatorname{Spec}(k) \to X$ represents p. Set S' to be the collection of $u \in U$ such that $(\mathbf{L}\pi^*E)_u \in \langle \mathcal{O}_{U,u} \rangle$. Applying Lemma 3.8, we see that this condition is independent of the choice of representative.

Let $q \in U$ satisfy $\pi(q) \in S$. Consider the natural morphisms

$$\operatorname{Spec}(\kappa(q)) \xrightarrow{i_q} \operatorname{Spec}(\mathcal{O}_{U,q}) \xrightarrow{t_q} U.$$

By hypothesis, $\mathbf{L}(\pi \circ t_q \circ i_q)^*E$ is perfect. This implies $\mathbf{L}(\pi \circ t_q)^*E \in \operatorname{Perf}(\mathcal{O}_{U,q})$ (see e.g. [GLMP25, Proposition 2.4]). Hence, $\pi^{-1}(S) \subseteq S'$.

Conversely, let $u \in S'$. Then $\mathbf{L}(\pi \circ t_u)^*E \in \operatorname{Perf}(\mathcal{O}_{U,u})$, and so $\mathbf{L}(\pi \circ t_u \circ i_u)^*E$ is perfect. Thus, $\pi^{-1}(S) = S'$.

Since π is surjective, it follows that $\pi(S') = S$. By [Let21, Proposition 2.5], S' is Zariski open in U (which can be checked on the small Zariski site [Sta26, Tags 071Q]). As π is a smooth morphism, it is an open morphism, and so S must be open. □

Theorem 3.43. *Let $f\colon Y \to X$ be a proper morphism of Noetherian algebraic spaces. For any $E \in D_{\operatorname{qc}}(Y)$ pseudocoherent, the following are equivalent:*

(1) E is f-perfect
(2) $E \otimes^{\mathbf{L}} \mathbf{L}f^*A \in D^b_{\operatorname{coh}}(Y)$ for all $A \in D^b_{\operatorname{coh}}(X)$
(3) E is f-quasi-perfect.

Proof. By Proposition 3.39, we have (1) $\iff$ (3). It is immediate that (1) $\implies$ (2). We prove (2) $\implies$ (3). Fix a closed point $p \in |X|$. Let $i\colon Z_p \to X$ be the residual space at p (see [Sta26, Tags 06QZ & 06R0]). By [Sta26, Tag 0H1R], Z_p can be represented by $\operatorname{Spec}(k)$ for some field k, i.e. we can choose $Z_p = \operatorname{Spec}(k)$. Fix any $Q \in \operatorname{Perf}(Y)$. Since E is pseudocoherent, (2) implies E has bounded cohomology. Hence, $\mathbf{R}f_*(E \otimes^{\mathbf{L}} Q) \in D^b_{\operatorname{coh}}(X)$. By (2), it follows that $E \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\operatorname{coh}}(Y)$. Since Q is perfect, we have that

$$Q \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\operatorname{coh}}(Y).$$

Moreover, f being proper implies

$$\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}f^*\mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)}) \in D^b_{\operatorname{coh}}(X).$$

By projection formula, we obtain

$$\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \in D^b_{\operatorname{coh}}(X).$$

A further application of projection tells us that

$$\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}} \mathbf{R}i_*\mathcal{O}_{\operatorname{Spec}(k)} \cong \mathbf{R}i_*\mathbf{L}i^*\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E).$$

Then Lemma 3.3 implies $\mathbf{L}i^*\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E) \in D^b_{\mathrm{coh}}(\mathrm{Spec}(k))$. By Lemma 3.42, the collection B of $p \in |X|$ such that

$$\mathbf{L}s^*(\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E))$$

is perfect, where $s\colon \mathrm{Spec}(k) \to X$ is a representative of p, is open in $|X|$. However, the work above shows every closed point belongs to this collection. By Remark 3.11, every point in $|X|$ is a generalization of a closed point, and so $B = |X|$. Therefore, Proposition 3.12 implies $\mathbf{R}f_*(Q \otimes^{\mathbf{L}} E) \in \mathrm{Perf}(X)$. □

Proposition 3.44. *Let S be a Noetherian algebraic space. Consider proper morphisms of algebraic spaces $f_1\colon Y_1 \to S$ and $f_2\colon Y_2 \to S$. Denote by $p_i\colon Y_1 \times_S Y_2 \to Y_i$ the natural projections. For any $E \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ pseudocoherent, the following are equivalent:*

1. *E is p_1-perfect*
2. *$\Phi_E(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$.*

Proof. First, assume E is p_1-perfect. Base change says each p_i is proper. It follows that

$$E \otimes^{\mathbf{L}} \mathbf{L}p_1^* D^b_{\mathrm{coh}}(Y_1) \subseteq D^b_{\mathrm{coh}}(Y_1 \times_S Y_2).$$

Since p_2 is proper, we have

$$\mathbf{R}(p_2)_*(E \otimes^{\mathbf{L}} \mathbf{L}p_1^* D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2).$$

See [Sta26, Tag 08GK]. We prove the converse.

Recall that each $D_{\mathrm{qc}}(Y_i)$ is compactly generated by an object G_i. Moreover, from Lemma B.1, $\mathbf{L}p_1^*G_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^*G_2$ is a compact generator for $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Choose $B \in D^b_{\mathrm{qc}}(Y_1)$. We want to show that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B \in D^b_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Applying Theorem 3.43, we test for p_1-perfectness on $B \in D^b_{\mathrm{coh}}(Y_1)$. Since E and $\mathbf{L}p_1^*B$ are pseudocoherent, it follows that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B$ is pseudocoherent, and so $E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B \in D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$. To show that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$, it suffices to check that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B \in D^+_{\mathrm{qc}}(Y_1 \times_S Y_2)$. By adjunctions,

$$\begin{aligned}
&\mathrm{Ext}^n(\mathbf{L}p_1^*G_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^*G_2, E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B) \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^*G_2, \mathbf{R}\mathcal{H}om(\mathbf{L}p_1^*G_1, E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B)) && \text{(tensor/hom)} \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^*G_2, \mathbf{R}\mathcal{H}om(\mathbf{L}p_1^*G_1, \mathcal{O}_{Y_1 \times_S Y_2}) \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B) && \text{([Sta26, Tag 08JJ])} \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^*G_2, \mathbf{R}\mathcal{H}om(\mathbf{L}p_1^*G_1, \mathbf{L}p_1^*\mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B) && (\mathbf{L}p_1^*\mathcal{O}_{Y_1} = \mathcal{O}_{Y_1 \times_S Y_2}) \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^*G_2, \mathbf{L}p_1^*(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1})) \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}p_1^*B) && \text{(Lemma 2.11)} \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^*G_2, \mathbf{L}p_1^*(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B) \otimes^{\mathbf{L}} E) && \text{([Sta26, Tag 07A4])} \\
&\cong \mathrm{Ext}^n(G_2, \mathbf{R}(p_2)_*(\mathbf{L}p_1^*(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B) \otimes^{\mathbf{L}} E)) && \text{(pull/push)} \\
&\cong \mathrm{Ext}^n(G_2, \Phi_E(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B)).
\end{aligned}$$

Clearly, $\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1})$ is perfect, and so

$$\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B \in D^b_{\mathrm{coh}}(Y_1).$$

Since

$$\Phi_E(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B) \in D^b_{\mathrm{coh}}(Y_2),$$

we know that

$$\Phi_E(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B) \in D^+_{\mathrm{qc}}(Y_2).$$

Hence, by Lemma 3.38, there exists an $i \gg 0$ such that for all $t \geq i$, one has

$$\operatorname{Hom}(G_2[t], \Phi_E(\mathbf{R}\mathcal{H}om(G_1, \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} B)) \cong 0.$$

Then the chain of isomorphisms above imply the claim via Lemma 3.38. □

Corollary 3.45. *Let S be a Noetherian algebraic space. Consider proper morphisms of algebraic spaces $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$. Denote by $p_i \colon Y_1 \times_S Y_2 \to Y_i$ the natural projections. For any $E \in D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$, the following are equivalent:*

(1) E is p_1-perfect (resp. p_2-perfect)

(2) $\Phi_E(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$ (resp. $\Phi_E(\operatorname{Perf}(Y_1)) \subseteq \operatorname{Perf}(Y_2)$).

Proof. This follows from Lemma 3.26, Proposition 3.44, and Theorem 3.43. □

4. Base change behavior

4.1. **Base change for relative perfectness.** We study the behavior of integral transforms under base change with respect to preserving D^b_{coh} and/or Perf.

Setup 4.1. Let $t \colon T \to S$ be a morphism of Noetherian algebraic spaces. Suppose $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ are proper flat morphisms of algebraic spaces. Consider the commutative diagram

(4.1)
$$\begin{tikzcd}
Y_1 \times_S Y_2 \times_S T \arrow[rr, "g_1'"] \arrow[dr, "g_2'"] \arrow[dd, "t'"'] & & Y_2 \times_S T \arrow[dr, "g_2"] \arrow[dd, dashed, "t_2" near end] & \\
& Y_1 \times_S T \arrow[rr, crossing over, "g_1" near start] & & T \arrow[dd, "t"] \\
Y_1 \times_S Y_2 \arrow[rr, dashed, "f_1'" near start] \arrow[dr, "f_2'"'] & & Y_2 \arrow[dr, "f_2"] & \\
& Y_1 \arrow[rr, "f_1"'] \arrow[from=uu, crossing over, "t_1" near start] & & S
\end{tikzcd}$$

obtained by base change along t.

Remark 4.2. Consider Setup 4.1 with t a quasi-affine morphism. By base change, t' and each t_i are quasi-affine. Moreover, base change implies each algebraic space in the cube is Noetherian. Note that $D_{\mathrm{qc}}(Y_1)$ and $D_{\mathrm{qc}}(Y_2)$ are respectively compactly generated by some G_1 and G_2. Hence, Lemma B.1 shows that $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ is compactly generated by $\mathbf{L}(f_2')^* G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* G_2$. Furthermore, from Lemma B.3, we have that $D_{\mathrm{qc}}(Y_1 \times_S T)$, $D_{\mathrm{qc}}(Y_2 \times_S T)$, and $D_{\mathrm{qc}}(Y_1 \times_S T \times_S Y_2)$ are respectively compactly generated by $\mathbf{L}t_1^* G_1$, $\mathbf{L}t_2^* G_2$, and $\mathbf{L}(t')^*(\mathbf{L}(f_2')^* G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* G_2)$.

Lemma 4.3. *Consider Setup 4.1. Then each face of (4.1) is a tor-independent square.*

Proof. This follows from the fact that f_i, f_i', g_i, and g_i' are flat. This can be checked after taking étale presentations, applying [Sta26, Tag 08IQ] and reducing to the scheme case (see e.g. [GW20, Lemma 22.93 & Remark 22.94]). □

Lemma 4.4. *Consider Setup 4.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Then on D_{qc} there is a natural isomorphism*

$$\beta^K \colon \mathbf{L}t_2^* \circ \Phi_K \to \Phi_{\mathbf{L}(t')^* K} \circ \mathbf{L}t_1^*.$$

Proof. There exists a string of natural isomorphisms for each $E \in D_{\mathrm{qc}}(Y_1)$:

$$\begin{aligned}
\mathbf{L}t_2^*\Phi_K(E) &= \mathbf{L}t_2^*\mathbf{R}(f_1')_*(\mathbf{L}(f_2')^*E \otimes^{\mathbf{L}} K) \\
&\cong \mathbf{R}(g_1')_*\mathbf{L}(t')^*(\mathbf{L}(f_2')^*E \otimes^{\mathbf{L}} K) && \text{(flat base change)} \\
&\cong \mathbf{R}(g_1')_*(\mathbf{L}(t')^*\mathbf{L}(f_2')^*E \otimes^{\mathbf{L}} \mathbf{L}(t')^*K) && \text{(monoidality)} \\
&\cong \mathbf{R}(g_1')_*(\mathbf{L}(g_2')^*\mathbf{L}(t_1)^*E \otimes^{\mathbf{L}} \mathbf{L}(t')^*K) && \text{(pseudofunctoriality)} \\
&= \Phi_{\mathbf{L}(t')^*K}(\mathbf{L}t_1^*E).
\end{aligned}$$

This completes the proof. □

Lemma 4.5. *Let $f\colon Y \to X$ be a proper flat morphism of Noetherian algebraic spaces. Choose a closed subset $Z \subseteq |Y|$. Then $\mathbf{R}f_*D^b_{\mathrm{coh},Z}(Y) \subseteq D^b_{\mathrm{coh},f(Z)}(X)$.*

Proof. For any $E \in D^b_{\mathrm{coh},Z}(Y)$, define $W_E := \mathrm{supp}(\mathbf{R}f_*E)$. Choose $p \in W_E$, and let $t\colon \mathrm{Spec}(k) \to X$ be a representative of p. Consider the fibered square

$$\begin{tikzcd}
Y \times_X \mathrm{Spec}(k) \arrow[r, "f'"] \arrow[d, "t'"'] & \mathrm{Spec}(k) \arrow[d, "t"] \\
Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

By Lemma 2.5, $\mathbf{L}t^*\mathbf{R}f_*E \ncong 0$. Applying flat base change, it follows that $\mathbf{R}f'_*\mathbf{L}(t')^*E \ncong 0$. This ensures that $Y \times_X \mathrm{Spec}(k)$ is nonempty (otherwise, $\mathbf{R}f'_*\mathbf{L}(t')^*E \cong 0$), and so f' is surjective. Moreover, it follows that $\mathbf{L}(t')^*E \ncong 0$. Choose any point $q \in \mathrm{supp}(\mathbf{L}(t')^*E)$, and representative $h\colon \mathrm{Spec}(\ell) \to Y \times_X \mathrm{Spec}(k)$ of q. By Lemma 2.5, $\mathbf{L}(t' \circ h)^*E \ncong 0$. Using Lemma 2.4, $(t' \circ h)$ represents $t'(q)$. By Lemma 2.5, $t'(q) \in \mathrm{supp}(E) \subseteq Z$. Therefore, $p = (f \circ t')(q) \in f(Z)$. □

Lemma 4.6. *Consider Setup 4.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$. For any $p \notin (f_1 \circ f_2')(\mathrm{supp}(K))$, and for any representative $t\colon \mathrm{Spec}(k) \to S$ of p, one has that $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S$ $\mathrm{Spec}(k)$ (resp. $Y_2 \times_S \mathrm{Spec}(k)$). In particular, $\Phi_{\mathbf{L}(t')^*K}(E) \cong 0$ for all $E \in D_{\mathrm{qc}}(Y_1 \times_S \mathrm{Spec}(k))$.*

Proof. By Lemma 4.5, $(f_1 \circ f_2')(\mathrm{supp}(K)) \subseteq |S|$ is closed. We claim that $\mathbf{L}(t')^*K \cong 0$ for all such p. Assume the contrary. Choose a representative $t\colon \mathrm{Spec}(k) \to S$ of $p \in |S|$ such that $\mathbf{L}(t')^*K \neq 0$. By Lemma 2.5, there exists $q \in |Y_1 \times_S Y_2 \times_S \mathrm{Spec}(k)|$ and a representative $a\colon \mathrm{Spec}(k') \to Y_1 \times_S Y_2 \times_S \mathrm{Spec}(k)$ of q such that $\mathbf{L}(t' \circ a)^*K \ncong 0$. Then Lemmas 2.4 and 2.5 imply $t'(q) \in \mathrm{supp}(K)$. However,

$$(f_1 \circ f_2' \circ t' \circ a)(q) = p \in (f_1 \circ f_2')(\mathrm{supp}(K)),$$

which is absurd. Therefore, $\mathbf{L}(t')^*K \cong 0$ for all $p \notin (f_1 \circ f_2')(\mathrm{supp}(K))$. In fact, the last claim becomes clear. □

Lemma 4.7. *Consider Setup 4.1 with t an affine morphism. Let $K \in D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$. If K is relatively perfect over Y_1 (resp. over Y_2), then $\Phi_{\mathbf{L}(t')^*K}$ restricts to an exact functor on D^b_{coh} (resp. on* Perf*).*

Proof. Suppose K is relatively perfect over Y_2. As $\Phi_K(\mathrm{Perf}(Y_1)) \subseteq \mathrm{Perf}(Y_2)$, Corollary 3.45 implies

$$\mathbf{R}(f_1')_*(K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2)) \subseteq \mathrm{Perf}(Y_2).$$

By base change, we know that t_1, t_2, t' are affine morphisms. Choose $G \in \operatorname{Perf}(Y_1 \times_S Y_2)$ such that $\operatorname{Perf}(Y_1 \times_S Y_2) = \langle G \rangle$. Now, $\mathbf{L}(t')^* G$ satisfies

$$\operatorname{Perf}(Y_1 \times_S Y_2 \times_S T) = \langle \mathbf{L}(t')^* G \rangle,$$

see [Sta26, Tag 09SR] and Remark 4.2. By base change, $g_1, g_2, f_2', f_1', g_2', g_1'$ are proper flat morphisms. Using flat base change, we have that

$$\begin{aligned} &\mathbf{R}(g_1')_*(\mathbf{L}(t')^* K \otimes^{\mathbf{L}} \operatorname{Perf}(Y_1 \times_S Y_2 \times_S T)) \\ &\quad \subseteq \mathbf{R}(g_1')_* \langle \mathbf{L}(t')^*(K \otimes^{\mathbf{L}} G) \rangle \\ &\quad \subseteq \langle \mathbf{R}(g_1')_* \mathbf{L}(t')^*(K \otimes^{\mathbf{L}} G) \rangle \\ &\quad \subseteq \langle \mathbf{L}t_2^* \mathbf{R}(f_1')_*(K \otimes^{\mathbf{L}} G) \rangle \\ &\quad \subseteq \operatorname{Perf}(Y_2 \times_S T). \end{aligned}$$

By Lemma 3.7, $\mathbf{L}(t')^* K$ is pseudocoherent. Thus, Corollary 3.45 says $\Phi_{\mathbf{L}(t')^* K}$ restricts to an exact functor on Perf. A similar argument proves the case for K being relatively perfect over Y_1. □

Proposition 4.8. *Consider Setup 4.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$.*

(1) If $\mathbf{L}(t')^* K$ is relatively perfect over $Y_1 \times_S T$ (resp. $Y_2 \times_S T$) where t is affine and faithfully flat, then K is relatively perfect over Y_1 (resp. Y_2).
(2) If $\mathbf{L}(t')^* K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$ (resp. $Y_2 \times_S \operatorname{Spec}(k)$) for all morphisms $t \colon \operatorname{Spec}(k) \to S$ from a field, then K is relatively perfect over Y_1 (resp. Y_2).
(3) If for every closed point $p \in |S|$ there exists a representative $t \colon \operatorname{Spec}(k) \to S$ of p such that $\mathbf{L}(t')^* K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$ (resp. $Y_2 \times_S \operatorname{Spec}(k)$), then K is relatively perfect over Y_1 (resp. Y_2).

Proof. We prove the case $\mathbf{L}(t')^* K$ is relatively perfect over $Y_1 \times_S T$. The other case is argued similarly using Lemma 3.7.

We check the first claim. Let $E \in D^b_{\mathrm{coh}}(Y_1)$. As t' is flat, it follows that $\mathbf{L}t_1^* E \in D^b_{\mathrm{coh}}(Y_1 \times_S T)$. By Lemma 4.4, we know that $\mathbf{L}t_2^* \Phi_K(E) \cong \Phi_{\mathbf{L}(t')^* K}(\mathbf{L}t_1^* E)$. If $\mathbf{L}(t')^* K$ is relatively perfect over $Y_1 \times_S T$, then Corollary 3.45 shows $\Phi_{\mathbf{L}(t')^* K}(\mathbf{L}t_1^* E) \in D^b_{\mathrm{coh}}(Y_2 \times_S T)$. Since t_2 is faithfully flat, we know for each $j \in \mathbb{Z}$ that $\mathcal{H}^j(\mathbf{L}t_2^* \Phi_K(E)) \cong 0$ implies $\mathcal{H}^j(\Phi_K(E)) \cong 0$ (see Lemma 2.10). Consequently, we have $\Phi_K(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$, which implies K is relatively perfect over Y_1 by Corollary 3.45.

Next, we check the second claim. Fix a compact generator $G \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. By Lemma 4.5,

$$\operatorname{supp}(\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G)) \subseteq f_2'(\operatorname{supp}(K \otimes^{\mathbf{L}} G)).$$

To show $\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G)$ is perfect, Proposition 3.12 says we can test for boundedness of the derived pullback of $\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G)$ along representatives of points in $|Y_1|$. By Lemma 2.5, it suffices to check this condition at all points in $\operatorname{supp}(\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G))$ since the derived pullback is the zero object otherwise. Choose any $p \in f_2'(\operatorname{supp}(K \otimes^{\mathbf{L}} G))$, and let $h_1 \colon \operatorname{Spec}(k) \to Y_1$ be a representative of p. Denote by $t \colon \operatorname{Spec}(k) \to S$ the composition

$f_1 \circ h_1$. There exists a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[dr, "h"] \arrow[drr, bend left, "1_{\mathrm{Spec}(k)}"] \arrow[ddr, bend right, "h_1"'] & & \\
& Y_1 \times_S \mathrm{Spec}(k) \arrow[r, "g_1"] \arrow[d, "t_1"'] & \mathrm{Spec}(k) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"] & S.
\end{tikzcd}$$

As $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \mathrm{Spec}(k)$, it follows that

$$\begin{aligned}
&\mathbf{R}(g_2')_*\mathbf{L}(t')^*(K \otimes^{\mathbf{L}} G) \\
&\quad\cong \mathbf{R}(g_2')_*(\mathbf{L}(t')^*K \otimes^{\mathbf{L}} \mathbf{L}(t')^*G) \\
&\quad\in \mathbf{R}(g_2')_*(\mathbf{L}(t')^*K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2 \times_S \mathrm{Spec}(k)) \\
&\quad\subseteq \mathrm{Perf}(Y_1 \times_S \mathrm{Spec}(k)).
\end{aligned}$$

There is a fibered square

$$\begin{tikzcd}
Y_1 \times_S Y_2 \times_S \mathrm{Spec}(k) \arrow[r, "g_2'"'] \arrow[d, "t'"'] & Y_1 \times_S \mathrm{Spec}(k) \arrow[d, "t_1"] \\
Y_1 \times_S Y_2 \arrow[r, "f_2'"'] & Y_1
\end{tikzcd}$$

By flat base change,

$$\begin{aligned}
\mathbf{L}t_1^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G) &\cong \mathbf{R}(g_2')_*\mathbf{L}(t')^*(K \otimes^{\mathbf{L}} G) \\
&\in \mathrm{Perf}(Y_1 \times_S \mathrm{Spec}(k)).
\end{aligned}$$

This implies that

$$\mathbf{L}h_1^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G) \cong \mathbf{L}h^*\mathbf{L}t_1^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G) \in D^b_{\mathrm{coh}}(k),$$

because the derived pullback of perfect complexes remains perfect. Thus, K is relatively perfect over Y_1.

Lastly, we check the third claim. Assume for every closed point $p \in |S|$ there exists a representative $t\colon \mathrm{Spec}(k) \to S$ of p such that $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \mathrm{Spec}(k)$. Fix a compact generator $G \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. It suffices to check that $\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G)$ is perfect. By Lemma 3.42, we only need to show that every closed point of $|Y_1|$ belongs to the collection of $p \in |Y_1|$ such that $\mathbf{L}s^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} G)$ is perfect, where $s\colon \mathrm{Spec}(k) \to Y_1$ is a representative of p. Choose a closed point $p \in |Y_1|$. Let $h_1\colon \mathrm{Spec}(k) \to Y_1$ be a representative of p. Denote by $h\colon \mathrm{Spec}(k) \to S$ the composition $f_1 \circ h_1$. Since f_1 is closed, $f_1(p) \in |S|$ is closed. By the hypotheses, there exists a representative $\tilde{t}\colon \mathrm{Spec}(k') \to S$ of $f_1(p)$ such that $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \mathrm{Spec}(k')$. Applying Lemma 2.4, h and $\tilde{t}$ represent $f_1(p)$. Hence, there exist a field ℓ and a commutative diagram,

$$\begin{tikzcd}
\mathrm{Spec}(\ell) \arrow[r, "a"] \arrow[d, "b"'] & \mathrm{Spec}(k') \arrow[d, "\tilde{t}"] \\
\mathrm{Spec}(k) \arrow[r, "h"'] & S.
\end{tikzcd}$$

The hypotheses says the derived pullback of K along $\tilde{t}$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k')$. As a is affine, Lemma 4.7 with Corollary 3.45 implies the derived pullback of K along $t \circ a$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(\ell)$. However, b is faithfully flat and affine, so the first claim (4.8(1)) implies the derived pullback of K along h is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$ (e.g. use that $t \circ a = h \circ b$). Consequently, one can now apply the argument from the second claim using h. □

Theorem 4.9. *Consider Setup 4.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Then the following are equivalent:*

(1) K is relatively perfect over Y_1 (resp. Y_2)
(2) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$ (resp. $Y_2 \times_S T$) for every affine morphism t from a Noetherian algebraic space
(3) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$ (resp. $Y_2 \times_S \operatorname{Spec}(k)$) for all morphisms $t\colon \operatorname{Spec}(k) \to S$ from a field
(4) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$ (resp. $Y_2 \times_S \operatorname{Spec}(k)$) for all morphisms $t\colon \operatorname{Spec}(k) \to S$ that represents a closed point $p \in |S|$
(5) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$ (resp. $Y_2 \times_S T$) for some affine faithfully flat morphism $t\colon T \to S$ of Noetherian algebraic spaces.

Proof. By Lemma 3.7, the derived pullback of K along any morphism above is pseudocoherent. We only prove the case of being relatively perfect over Y_1, since the other follows analogously. Applying Lemma 4.7 with Corollary 3.45, (1) implies (2). Moreover, (2) implies (3) because S is quasi-separated [Sta26, Tag 09TF]. Clearly, (3) implies (4). Furthermore, to see that (4) implies (5), note that Proposition 4.8(3) shows that K is relatively perfect over Y_1, i.e. take $t = 1_S$. Lastly, by Proposition 4.8(1), (5) implies (1). □

Example 4.10. We show that Theorem 4.9 is sharp. Let $X = \operatorname{Spec}(\mathbb{Z})$. Choose an infinite sequence of prime numbers $p_1 < p_2 < \cdots$. Denote by $t_n\colon \operatorname{Spec}(\mathbf{F}_{p_n}) \to X$ the associated closed immersion. Set $K := \oplus_{n\geq 1}(t_n)_*\mathcal{O}_{\operatorname{Spec}(\mathbf{F}_{p_n})}[n]$. Note that $K \in D^-_{\mathrm{coh}}(X)$. Moreover, $\Phi_{\mathbf{L}(t'_n)^*K}$ induces an endofunctor on $D^b_{\mathrm{coh}}(\operatorname{Spec}(\mathbf{F}_{p_n}))$ for all n (i.e. Setup 4.1 with $t := t_n$ and $f_i = 1_{\mathbb{Z}}$). However, $\Phi_K(\mathcal{O}_{\operatorname{Spec}(\mathbb{Z})})$ has infinite cohomology.

4.2. **Derived reflexivity**. We introduce an analog of [AIL11, Definition 1.3.1].

Definition 4.11. Let X be an algebraic space and $E, A \in D(X)$. We say that E is **derived A-reflexive** if the following hold:

(1) the natural morphism
$$E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, A), A)$$
is an isomorphism
(2) E and $\mathbb{R}\mathcal{H}om(E, A)$ are bounded pseudocoherent complexes.

Remark 4.12. Definition 4.11 is an analog of [AIL11, Definition 1.3.1]. For convenience, we recall that the natural morphism is obtained via the following sequence of adjunctions:

$$\begin{aligned}
\operatorname{Hom}(\mathbb{R}\mathcal{H}om(E, A), \mathbb{R}\mathcal{H}om(E, A)) &\cong \operatorname{Hom}(\mathbb{R}\mathcal{H}om(E, A) \otimes^{\mathbf{L}} E, A) \\
&\cong \operatorname{Hom}(E \otimes^{\mathbf{L}} \mathbb{R}\mathcal{H}om(E, A), A) \\
&\cong \operatorname{Hom}(E, \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, A), A)).
\end{aligned}$$

Take the morphism obtained from the identity of $\mathbb{R}\mathcal{H}om(E, A)$.

Lemma 4.13. *Let X be a Noetherian algebraic space. Consider $E, A \in D(X)$ such that E and $\mathbb{R}\mathcal{H}om(E,A)$ are bounded pseudocoherent complexes. Then the following are equivalent:*

(1) E is derived A-reflexive

(2) there exists an isomorphism

$$E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,A),A)$$

(3) $\mathbf{L}s^*E$ is derived $\mathbf{L}s^*A$-reflexive for every étale morphism $s\colon Y \to X$ from a scheme

(4) there exists an étale presentation $s\colon Y \to X$ such that $\mathbf{L}s^*E$ is derived $\mathbf{L}s^*A$-reflexive.

Moreover, if A and B are isomorphic objects of $D(X)$, then E is derived A-reflexive if, and only if, it is derived B-reflexive.

Proof. It is immediate that (1) $\implies$ (2). We prove (2) $\implies$ (1). Suppose there exists an isomorphism

$$\alpha\colon E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,A),A).$$

Choose an étale presentation $s\colon U \to X$. Then it follows that $\mathbf{L}s^*E$ is isomorphic to $\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(\mathbf{L}s^*E,\mathbf{L}s^*A),\mathbf{L}s^*A)$ [Sta26, Tags 04LX & 08JB]. By [AIL11, Proposition 1.3.3], the natural morphism

$$\mathbf{L}s^*E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(\mathbf{L}s^*E,\mathbf{L}s^*A),\mathbf{L}s^*A)$$

is an isomorphism, and so,

$$\mathbf{L}s^*\operatorname{cone}(E \xrightarrow{ntrl.} \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,A),A)) \cong 0.$$

Hence, Lemma 2.9 implies

$$\operatorname{cone}(E \xrightarrow{ntrl.} \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,A),A)) \cong 0,$$

and so E is derived A-reflexive.

To see that (2) $\implies$ (3), pullback the isomorphism, use [Sta26, Tags 04LX & 08JB], and apply (2) on Y. Clearly, (3) $\implies$ (4), whereas (4) $\implies$ (1) is argued like above.

Finally, we prove the last claim. Let $\alpha\colon A \to B$ be an isomorphism. If E is derived A-reflexive, then the natural morphism

$$E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,A),A)$$

is an isomorphism. Moreover, α induces an isomorphism

$$\mathbb{R}\mathcal{H}om(E,A) \xrightarrow{\mathbb{R}\mathcal{H}om(E,\alpha)} \mathbb{R}\mathcal{H}om(E,B),$$

and hence, an isomorphism

$$\mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,A),A) \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,B),B).$$

Consequently, we obtain an isomorphism

$$E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E,B),B).$$

By (1), E is derived B-reflexive. The same argument gives the converse. □

Lemma 4.14. *Let $f\colon Y \to X$ be a separated finitely presented morphism of Noetherian algebraic spaces. Then $f^!(D^+_{\mathrm{qc}}(X)) \subseteq D^+_{\mathrm{qc}}(Y)$.*

Proof. Let $E \in D^+_{\mathrm{qc}}(X)$. Choose an étale presentation $s\colon U \to X$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

By Lemmas 2.12 and 2.13, s is separated. Base change implies each morphism in the diagram is finitely presented, and each algebraic space is Noetherian. By Lemma C.3, we have

$$\mathbf{L}(s')^* f^! E \cong (f')^! \mathbf{L}s^* E.$$

Observe that $\mathbf{L}s^*E \in D^+_{\mathrm{qc}}(U)$. Choose an étale surjective morphism $t\colon V \to Y \times_X U$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd} V' \arrow[r, "t_2"] \arrow[d, "t_1"'] & V \arrow[d, "t"] \\ V \arrow[r, "t"'] & Y \times_X U. \end{tikzcd}$$

By Lemma 2.13, t is quasi-affine. In particular, t is representable by schemes. As V is affine, each t_i is separated and étale with scheme source.

Applying Lemma C.3, there exists an isomorphism

$$t_2^! \mathbf{L}t^*(f')^! \mathbf{L}s^* E \cong \mathbf{L}t_1^* t^! (f')^! \mathbf{L}s^* E.$$

Since t_i is étale, [Sta26, Tag 0FWI] shows that $t_i^! \cong \mathbf{L}t_i^*$. Combining gives the identifications,

$$\mathbf{L}t_2^* \mathbf{L}t^*(f')^! \mathbf{L}s^* E \cong t_2^! \mathbf{L}t^*(f')^! \mathbf{L}s^* E \cong \mathbf{L}t_1^* t^! (f')^! \mathbf{L}s^* E.$$

Furthermore, from Lemma C.1, $(f' \circ t)^! \mathbf{L}s^* E \cong t^!(f')^! \mathbf{L}s^* E$. By [Sta26, Tag 0A9Y], it follows that $(f' \circ t)^! \mathbf{L}s^* E \in D^+_{\mathrm{qc}}(V)$, and hence,

$$\mathbf{L}t_2^* \mathbf{L}t^*(f')^! \mathbf{L}s^* E \cong \mathbf{L}t_1^* t^! (f')^! \mathbf{L}s^* E \in D^+_{\mathrm{qc}}(V').$$

Since $t \circ t_2$ is an étale presentation, Lemma 2.9 shows that $(f')^! \mathbf{L}s^* E \in D^+_{\mathrm{qc}}(Y \times_X U)$. Similar reasoning shows $f^! E \in D^+_{\mathrm{qc}}(Y)$ as desired. □

Lemma 4.15. *Let $f\colon Y \to X$ be a separated finitely presented morphism of Noetherian algebraic spaces. Assume X has affine diagonal. If E is an f-perfect pseudocoherent complex on Y, then there exists an isomorphism*

$$E \to \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^!\mathcal{O}_X), f^!\mathcal{O}_X).$$

In fact, E is derived $f^!\mathcal{O}_X$-reflexive.

Proof. We show that E is derived $f^!\mathcal{O}_X$-reflexive. Choose an étale presentation $s\colon U \to X$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Note that s is separated Lemmas 2.12 and 2.13. Base change implies each morphism in the diagram is finitely presented, and each algebraic space is Noetherian. By Lemma 4.13, it suffices to show that $\mathbf{L}(s')^*E$ is derived $\mathbf{L}(s')^*f^!\mathcal{O}_X$-reflexive. By Lemma C.3, we have

$$\mathbf{L}(s')^*f^!\mathcal{O}_X \cong (f')^!\mathbf{L}s^*\mathcal{O}_X \cong (f')^!\mathcal{O}_U.$$

Thus, again from Lemmas 2.12 and 2.13, it suffices to check that $\mathbf{L}(s')^*E$ is derived $(f')^!\mathcal{O}_U$-reflexive.

Choose an étale presentation $t\colon V \to Y \times_X U$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd} V' \arrow[r, "t_2"] \arrow[d, "t_1"'] & V \arrow[d, "t"] \\ V \arrow[r, "t"'] & Y \times_X U. \end{tikzcd}$$

By Lemma 2.13, t is quasi-affine. In particular, t is representable by schemes. As V is affine, each t_i is separated and étale with scheme source. Applying Lemma C.3, there exists an isomorphism

$$t_2^!\mathbf{L}t^*(f')^!\mathcal{O}_U \cong \mathbf{L}t_1^*t^!(f')^!\mathcal{O}_U.$$

Since t_i is étale, [Sta26, Tag 0FWI] shows that $t_i^! \cong \mathbf{L}t_i^*$. Combining gives the identifications,

$$\mathbf{L}t_2^*\mathbf{L}t^*(f')^!\mathcal{O}_U \cong t_2^!\mathbf{L}t^*(f')^!\mathcal{O}_U \cong \mathbf{L}t_1^*t^!(f')^!\mathcal{O}_U. \tag{4.2}$$

As X has affine diagonal, s is affine [Sta26, Tag 08GB]. Then, by Proposition 3.20, $\mathbf{L}(s')^*E$ is f'-perfect because E is f-perfect. Moreover, flatness of t implies $\mathbf{L}(s' \circ t)^*E$ is $(f' \circ t)$-perfect. Furthermore, $\mathbf{L}(s' \circ t)^*E$ is pseudocoherent and $(f' \circ t)^!\mathcal{O}_U \in D_{\mathrm{qc}}(V)$. It follows from [Ill71, Corollaire 4.9.2] and Proposition 3.16 that $\mathbf{L}(s' \circ t)^*E$ is $(f' \circ t)^!\mathcal{O}_U$-reflexive. By Lemma C.1, $(f' \circ t)^!\mathcal{O}_U \cong t^!(f')^!\mathcal{O}_U$. As t_1 is flat, we have $\mathbf{L}(s' \circ t \circ t_1)^*E$ is $\mathbf{L}t_1^*(f' \circ t)^!\mathcal{O}_U$-reflexive. By (4.2), we obtain $\mathbf{L}(s' \circ t \circ t_1)^*E$ is $\mathbf{L}t_2^*\mathbf{L}t^*(f')^!\mathcal{O}_U$-reflexive. Since $t \circ t_2$ is an étale presentation, Lemma 2.9 implies $\mathbf{L}(s')^*E$ is $(f')^!\mathcal{O}_U$-reflexive.

Now, we have checked that E is derived $f^!\mathcal{O}_X$-reflexive. Hence, the natural morphism

$$E \to \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, f^!\mathcal{O}_X), f^!\mathcal{O}_X)$$

is an isomorphism. Moreover, E and $\mathbb{R}\mathcal{H}om(E, f^!\mathcal{O}_X)$ are bounded pseudocoherent complexes. In this situation, Lemma 2.7 yields an isomorphism,

$$\begin{aligned} E &\cong \mathbb{R}\mathcal{H}om(\mathbb{R}\mathcal{H}om(E, f^!\mathcal{O}_X), f^!\mathcal{O}_X) \\ &\cong \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(E, f^!\mathcal{O}_X), f^!\mathcal{O}_X). \end{aligned}$$

Thus, we complete the proof. □

4.3. **Base change for adjoints**.

We study the existence of adjoints and the behavior of adjointness for integral transforms under base change.

Lemma 4.16 (Base change for relative dualizing complex). *Consider a fibered square of Noetherian algebraic spaces*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & S' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where f is proper and flat. Then there is an isomorphism

$$\mathbf{L}(g')^* f^! \mathcal{O}_X \to (f')^! \mathbf{L} g^* \mathcal{O}_X.$$

Proof. It suffices by [Sta26, Tags 0E6C & 0E5Z] to show $f^! \mathcal{O}_X$ is relative dualizing complex for $f\colon Y \to X$ a flat proper morphism of Noetherian algebraic spaces. Choose étale presentation $s\colon U \to X$ from an affine scheme. Consider the fibered square

$$\begin{tikzcd} Y \times_X U \arrow[r, "f'"] \arrow[d, "s'"'] & U \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

We want to apply [Sta26, Tag 0E60]. By Lemma C.2, $f^! \cong f^\times$ and same for f'. Moreover, from Lemma C.3, we have isomorphism

$$(f')^! \mathcal{O}_U \cong (f')^! \mathbf{L} s^* \mathcal{O}_X \cong \mathbf{L}(s')^* f^! \mathcal{O}_X.$$

Applying [Sta26, Tag 0E5W] (see (3)), we see that $(f')^! \mathcal{O}_U$ is a dualizing complex with counit (i.e. the trace) $\mathbf{R} f'_*(f')^! \mathcal{O}_U \to \mathcal{O}_U$. In this setting, the formation of the trace map commutes with base change [Sta26, Tag 0E5K]. This completes the proof. □

Proposition 4.17. *Consider Setup 4.1 with t affine. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over Y_2. Then $\Phi_{\mathbf{L}(t')^* K'}$ is right adjoint to $\Phi_{\mathbf{L}(t')^* K}$ on D_{qc} where*

$$K' := \mathbf{R}\mathcal{H}om\big(K, (f_1')^! \mathcal{O}_{Y_2}\big).$$

In particular, the kernel of the right adjoint is a bounded pseudocoherent complex.

Proof. We check the first claim. By Lemma C.2, we have $b^\times \cong b^!$ on D_{qc} for any proper morphisms b of Noetherian algebraic spaces. For any $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$ and $A \in D_{\mathrm{qc}}(Y_2 \times_S T)$, there is a string of natural isomorphisms obtained by adjunctions:

$$\begin{aligned} &\mathrm{Hom}\big(\Phi_{\mathbf{L}(t')^*K}(E), A\big) \\ &\quad = \mathrm{Hom}\big(\mathbf{R}(g_1')_*(\mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} \mathbf{L}(t')^* K), A\big) \\ &\quad \cong \mathrm{Hom}\big(\mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} \mathbf{L}(t')^* K, (g_1')^! A\big) && \text{(right adjoint of push)} \\ &\quad \cong \mathrm{Hom}\big(\mathbf{L}(g_2')^* E, \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! A)\big) && \text{(tensor/hom)} \\ &\quad \cong \mathrm{Hom}\big(E, \mathbf{R}(g_2')_* \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! A)\big) && \text{(push/pull)}. \end{aligned}$$

The desired claim follows if we can find an isomorphism

$$\begin{aligned} &\mathbf{R}(g_2')_* \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! A) \\ &\qquad \cong \mathbf{R}(g_2')_* \Big(\mathbf{L}(t')^* \mathbf{R}\mathcal{H}om(K, (f_1')^! \mathcal{O}_{Y_2}) \otimes^{\mathbf{L}} \mathbf{L}(g_1')^* A\Big). \end{aligned}$$

Moreover, by Lemma 3.34, $(f_1')^! \mathcal{O}_{Y_2} \in D^+_{\mathrm{coh}}(Y_1 \times_S Y_2)$, and so

$$\begin{aligned} \mathbf{L}(t')^* \mathbf{R}\mathcal{H}om(K, (f_1')^! \mathcal{O}_{Y_2}) &\cong \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, \mathbf{L}(t')^* (f_1')^! \mathcal{O}_{Y_2}) && \text{(Lemma 2.11)} \\ &\cong \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! \mathcal{O}_{Y_2 \times_S T}) && \text{(Lemma 4.16)}. \end{aligned} \tag{4.3}$$

From Lemma 4.7, $\Phi_{\mathbf{L}(t')^* K}$ restricts to an exact functor

$$\mathrm{Perf}(Y_1 \times_S T) \to \mathrm{Perf}(Y_2 \times_S T).$$

Then Corollary 3.45 implies

$$\mathbf{R}(g_1')_*(\mathbf{L}(t')^*K \otimes^{\mathbf{L}} \operatorname{Perf}(Y_1 \times_S Y_2 \times_S T)) \subseteq \operatorname{Perf}(Y_2 \times_S T).$$

There is another string of isomorphisms for all $A \in D_{\mathrm{qc}}$:

$$\begin{aligned}
&\mathbf{R}(g_2')_* \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K, (g_1')^! A) \\
&\quad \cong \mathbf{R}(g_2')_* \Big(\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K, (g_1')^! \mathcal{O}_{Y_2 \times_S T}) \otimes^{\mathbf{L}} \mathbf{L}(g_1')^* A \Big) \quad (\text{Proposition 3.32}) \\
&\quad \cong \mathbf{R}(g_2')_* \Big(\mathbf{L}(t')^* \mathbf{R}\mathcal{H}om(K, (f_1')^! \mathcal{O}_{Y_2}) \otimes^{\mathbf{L}} \mathbf{L}(g_1')^* A \Big) \qquad (\text{use } (4.3)).
\end{aligned}$$

In particular, the second uses the fact that $\mathbf{L}(t')^*K$ is g_1'-quasi-perfect via Theorem 4.9 and Theorem 3.43. Lastly, the second claim follows from Lemmas 3.25 and 3.34. □

Corollary 4.18. *Consider Setup 4.1 with t affine. Suppose $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ is pseudocoherent and relatively perfect over each Y_i. Denote by K' the kernel of the integral transform obtained in Proposition 4.17 which is right adjoint to Φ_K on D_{qc}. Then $\Phi_{K'}$ restricts to D^b_{coh}. In particular, Φ_K and $\Phi_{K'}$ form an adjoint pair on D^b_{coh}.*

Proof. Since K is p_2-quasi-perfect, Lemma 3.30 implies that K' is p_2-quasi-perfect. Hence, by Corollary 3.45, $\Phi_{K'}(D^b_{\mathrm{coh}}(Y_2)) \subseteq D^b_{\mathrm{coh}}(Y_1)$. □

Lemma 4.19. *Consider Setup 4.1. Suppose $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ is pseudocoherent and relatively perfect over Y_1. Then Φ_K admits a left adjoint on D_{qc}. In particular, the left adjoint is of the form $\Phi_{K'}$ where $K' := \mathbf{R}\mathcal{H}om(K, (f_2')^! \mathcal{O}_{Y_1})$.*

Proof. By Lemma C.2, $(f_i')^\times \cong (f_i')^!$ for each i. Also, from Corollary 3.45, K is f_2'-quasi-perfect. Thus, the desired claim follows from the string of natural isomorphisms for all $E \in D_{\mathrm{qc}}(Y_2)$ and $G \in D_{\mathrm{qc}}(Y_1)$:

$$\begin{aligned}
\operatorname{Hom}(E, \Phi_K(G)) &\cong \operatorname{Hom}(E, \mathbf{R}(f_1')_*(K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^* G)) && (\text{definition}) \\
&\cong \operatorname{Hom}(\mathbf{L}(f_1')^* E, K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^* G) && (\text{adjunction}) \\
&\cong \operatorname{Hom}(\mathbf{R}(f_2')_*(\mathbf{R}\mathcal{H}om(K, (f_2')^! \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* E), G) && (\text{Theorem 3.37}).
\end{aligned}$$

□

Proposition 4.20. *Consider Setup 4.1 with t affine. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over Y_1. Then $\Phi_{\mathbf{L}(t')^*K'}$ is left adjoint to $\Phi_{\mathbf{L}(t')^*K}$ on D_{qc} where $K' := \mathbf{R}\mathcal{H}om(K, (f_2')^! \mathcal{O}_{Y_1})$. In such a case, $K' \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$.*

Proof. The second claim follows from Lemmas 3.25 and 3.34. We check the first claim. By Lemma C.2, we have $h^\times \cong h^!$ on D_{qc} for any proper morphism h of Noetherian algebraic spaces. By Theorem 4.9, $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$. Moreover, from Lemma 4.19, the kernel of the integral transform which is left adjoint to $\Phi_{\mathbf{L}(t')^*K}$ is given by the object $\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K, (g_2')^! \mathcal{O}_{Y_1 \times_S T})$. It suffices to show there is an isomorphism

$$\begin{aligned}
&\mathbf{R}(g_2')_*(\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K, (g_2')^! \mathcal{O}_{Y_1 \times_S T}) \otimes^{\mathbf{L}} \mathbf{L}(g_1')^* E) \\
&\qquad \to \mathbf{R}(g_2')_*(\mathbf{L}(t')^* \mathbf{R}\mathcal{H}om(K, (f_2')^! \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} \mathbf{L}(g_1')^* E).
\end{aligned}$$

This follows if we can find an isomorphism

$$\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K, (g_2')^! \mathcal{O}_{Y_1 \times_S T}) \to \mathbf{L}(t')^*(\mathbf{R}\mathcal{H}om(K, (f_2')^! \mathcal{O}_{Y_1})).$$

By Lemma 3.34, $(g_2')^! \mathcal{O}_{Y_1 \times_S T} \in D^+_{\mathrm{coh}}(Y_1 \times_S Y_2 \times_S T)$. Hence, we have a string of isomorphisms

$$\begin{aligned}
&\mathbf{L}(t')^* \mathbf{R}\mathscr{H}om(K, (f_2')^! \mathcal{O}_{Y_1}) \\
&\quad \cong \mathbf{R}\mathscr{H}om(\mathbf{L}(t')^* K, \mathbf{L}(t')^* (f_2')^! \mathcal{O}_{Y_1}) \quad (\text{Lemma 2.11}) \\
&\quad \cong \mathbf{R}\mathscr{H}om(\mathbf{L}(t')^* K, (g_2')^! \mathcal{O}_{Y_1 \times_S T}) \quad (\text{Lemma 4.16}).
\end{aligned}$$

This completes the proof. □

4.4. **Fully faithfulness & equivalences**. We study the behavior of full faithfulness or equivalences for integral transforms under base change. Much of this section follows similarly to [GLMP25, §4] due to our work in earlier sections. The arguments of loc. cit. rely on the following:

- flat base change and projection formula
- compatibility of adjoint pairs under base change
- the ability to detect full faithfulness via the units and counits of adjunctions.

In the setting of algebraic spaces, these ingredients are provided by:

- Proposition 4.17 and Corollary 4.18 (existence and control of adjoints)
- Lemmas 4.4 and 4.22 (base change formulas)
- Theorem 4.9 (descent and boundedness).

Consequently, most proofs in loc. cit. carry over after replacing the scheme theoretic inputs with the above results. However, for the sake of self-containedness, we make this concrete.

Remark 4.21. Those familiar with [AL12] might think the argument of loc. cit. applies for nonperfect kernels. This is not possible: [AL12, Lemma 2.3] is used to prove [AL12, Theorem 3.1], but it explicitly appears in the proof of Appendix A, Eq. (A.18) that perfectness of the kernel is required.

Lemma 4.22. *Consider Setup 4.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Then on D_{qc} there is a natural isomorphism*

$$\alpha^K : \Phi_K \circ \mathbf{R}(t_1)_* \to \mathbf{R}(t_2)_* \circ \Phi_{\mathbf{L}(t')^* K}.$$

Proof. This follows from the string of natural isomorphisms for each $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$:

$$\begin{aligned}
\Phi_K \circ \mathbf{R}(t_1)_*(E) &= \mathbf{R}(f_1')_*\big(\mathbf{L}(f_2')^* \mathbf{R}(t_1)_* E \otimes^{\mathbf{L}} K\big) \\
&\cong \mathbf{R}(f_1')_*\big(\mathbf{R}t'_* \mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} K\big) && (\text{flat base change}) \\
&\cong \mathbf{R}(f_1')_* \mathbf{R}t'_*\big(\mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} \mathbf{L}(t')^* K\big) && (\text{projection formula}) \\
&\cong \mathbf{R}(t_2)_* \mathbf{R}(g_1')_*\big(\mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} \mathbf{L}(t')^* K\big) && (\text{pseudofunctoriality}) \\
&= \mathbf{R}(t_2)_* \circ \Phi_{\mathbf{L}(t')^* K}(E).
\end{aligned}$$

We conclude the proof. □

Lemma 4.23. *Consider Setup 4.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Then we have a morphism of adjoint pairs as depicted in the diagram*

$$\begin{tikzcd}
D_{\mathrm{qc}}(Y_1) \arrow[r, "\Phi_K"] \arrow[d, shift right, "\mathbf{L}t_1^*"'] & D_{\mathrm{qc}}(Y_2) \arrow[d, shift right, "\mathbf{L}t_2^*"'] \\
D_{\mathrm{qc}}(Y_1 \times_S T) \arrow[r, "\Phi_{\mathbf{L}(t')^* K}"'] \arrow[u, shift right, "\mathbf{R}(t_1)_*"'] \arrow[u, phantom, "\dashv"] & D_{\mathrm{qc}}(Y_2 \times_S T) \arrow[u, shift right, "\mathbf{R}(t_2)_*"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

with left and right comparison transformations given respectively by the isomorphisms β^K, α^K of Lemmas 4.4 and 4.22.

Proof. This is argued verbatim to [GLMP25, Lemma 4.5]. The point is that one needs to check that certain diagrams commute using flat base change isomorphisms, projection formula (including its definition), and functoriality/monoidality of derived pullback and pushforward functors in Appendix C. Using Lemma 4.22, the argument of [GLMP25, Lemma 4.5] is readily adaptable. Particularly, the new ingredients required are Lemmas 4.4 and 4.22. We spell this out more carefully.

Denote by $\xi^i \colon \mathbf{L}t_i^*\mathbf{R}(t_i)_* \to \mathrm{id}$ for the counit of the adjunction $\mathbf{L}t_i^* \dashv \mathbf{R}(t_i)_*$ with $i = 1,2$. To prove the desired claim, it suffices to verify that [GLMP25, Lemma 4.1(1)] holds; that is, that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{L}t_2^*\Phi_K\mathbf{R}(t_1)_* \arrow[r, "\beta^K_{\mathbf{R}(t_1)_*}"] \arrow[d, "\mathbf{L}t_2^*(\alpha^K)"'] & \Phi_{\mathbf{L}(t')^*K}\mathbf{L}t_1^*\mathbf{R}(t_1)_* \arrow[d, "\Phi_{\mathbf{L}(t')^*K}(\xi^1)"] \\
\mathbf{L}t_2^*\mathbf{R}(t_2)_*\Phi_{\mathbf{L}(t')^*K} \arrow[r, "\xi^2_{\Phi_{\mathbf{L}(t')^*K}}"'] & \Phi_{\mathbf{L}(t')^*K}.
\end{tikzcd}$$

for brevity, in the remainder of the proof, we drop the $\mathbf{L}$'s and $\mathbf{R}$'s in the notation for the derived functors (i.e. all functors now are understood to be derived). Hence, the diagram above is now

$$\begin{tikzcd}
 & t_2^*(t_2)_*\Phi_{(t')^*K} \arrow[dr, "\xi^2_{\Phi_{(t')^*K}}"] & \\
t_2^*\Phi_K(t_1)_* \arrow[ur, "t_2^*(\alpha^K)"] \arrow[dr, "\beta^K_{(t_1)_*}"'] & & \Phi_{(t')^*K}. \\
 & \Phi_{(t')^*K}t_1^*(t_1)^* \arrow[ur, "\Phi_{(t')^*K}(\xi^1)"'] &
\end{tikzcd} \tag{4.4}$$

Choose $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$. If we expand the definition of α^K and β^K in (4.4) (see the proofs of Lemmas 4.4 and 4.22), we obtain a large diagram consisting of various faces. In what follows, we make this a bit more explicit. Particularly, we spell out each face needed to understand (4.4). To describe said diagram more explicitly, we use the following abbreviations for the (natural) isomorphisms:

- BC_i, $i \in \{1,2\}$ for the flat base change isomorphisms
- PF for the projection formula
- $F_{(-)_*}$ for functoriality of $(-)_*$
- $F_{(-)^*}$ for functoriality of $(-)^*$
- $M_{(-)^*}$ for monoidality of $(-)^*$
- ξ' for the counit of $(t')^* \dashv (t')_*$.

Now, consider the following diagrams:

$$\begin{tikzcd}
t_2^*(f_1')_*((f_2')_*(t_1)_*E \otimes K) \arrow[r, "\mathrm{BC}_1"] \arrow[d, "\mathrm{BC}_2"] & t_2^*(f_1')_*((t')_*(g_2')^*E \otimes K) \arrow[d, "\mathrm{BC}_2"] \\
(g_1')_*(t')^*((f_2')^*(t_1)_*E \otimes K) \arrow[r, "\mathrm{BC}_1"] & (g_1')_*(t')^*(t'_*(g_2')^*E \otimes K)
\end{tikzcd} \tag{4.5}$$

$$\begin{tikzcd}
t_2^*(f_1')_*((t')_*(g_2')^*E \otimes K) \arrow[r, "\mathrm{PF}"] \arrow[d, "\mathrm{BC}_2"] & t_2^*(f_1')_*(t')_*((g_2')^*E \otimes (t')^*K) \arrow[d, "\mathrm{BC}_2"] \\
(g_1')_*(t')^*(t'_*(g_2')^*E \otimes K) \arrow[r, "\mathrm{PF}"] & (g_1')_*(t')^*t'_*((g_2')^*E \otimes (t')^*K)
\end{tikzcd} \tag{4.6}$$

$$\begin{tikzcd}
(g_1')_*(t')^*((f_2')^*(t_1)_*E \otimes K) \arrow[r, "\mathrm{BC}_1"] \arrow[d, "M_{(-)^*}"] & (g_1')_*(t')^*(t'_*(g_2')^*E \otimes K) \arrow[d, "M^*_{(-)}"] \\
(g_1')_*((t')^*(f_2')^*(t_1)_*E \otimes (t')^*K) \arrow[r, "\mathrm{BC}_1"] & (g_1')_*((t')^*t'_*(g_2')^*E \otimes (t')^*K)
\end{tikzcd} \tag{4.7}$$

$$\begin{tikzcd}
(g_1')_*((t')^*t'_*(g_2')^*E \otimes (t')^*K) \arrow[d, "\xi'"] & (g_1')_*(t')^*(t'_*(g_2')^*E \otimes K) \arrow[l, "M^*_{(-)}"'] \arrow[d, "\mathrm{PF}"] \\
(g_1')_*((g_2')^*E \otimes (t')^*K) & (g_1')_*(t')^*t'_*((g_2')^*E \otimes (t')^*K) \arrow[l, "\xi'"']
\end{tikzcd} \tag{4.8}$$

$$\begin{tikzcd}
t_2^*(f_1')_*(t')_*((g_2')^*E \otimes (t')^*K) \arrow[r, "F_{(-)^*}"] \arrow[d, "\mathrm{BC}_2"] & t_2^*(t_2)_*(g_1')_*((g_2')^*E \otimes (t')^*K) \arrow[d, "\xi^2"] \\
(g_1')_*(t')^*t'_*((g_2')^*E \otimes (t')^*K) \arrow[r, "\xi'"] & (g_1')_*((g_2')^*E \otimes (t')^*K)
\end{tikzcd} \tag{4.9}$$

$$\begin{tikzcd}
(g_1')_*((t')^*(f_2')^*(t_1)_*E \otimes (t')^*K) \arrow[r, "\mathrm{BC}_1"] \arrow[d, "F_{(-)^*}"] & (g_1')_*((t')^*t'_*(g_2')^*E \otimes (t')^*K) \arrow[d, "\xi'"] \\
(g_1')_*((g_2')^*t_1^*(t_1)_*E \otimes (t')^*K) \arrow[r, "\xi^1"] & (g_1')_*((g_2')^*E \otimes (t')^*K)
\end{tikzcd} \tag{4.10}$$

Assume that we have shown (4.5), (4.6), (4.7), (4.8), (4.9), (4.10) are commutative. Then (4.4) (in the case of E) is the pasting of these diagrams, i.e. gives the desired morphism

$$t_2^*(f_1')_*((f_2')_*(t_1)_*E \otimes K) \longrightarrow (g_1')_*((g_2')^*E \otimes (t')^*K).$$

So, let us explain why (4.5), (4.6), (4.7), (4.8), (4.9), (4.10) are commutative. As for (4.5), one may use naturality of BC_2, whereas (4.6) is due to the naturality of BC_2. Also, for (4.7), one can use the naturality of $M_{(-)^*}$.

Next, we explain (4.8). Observe that it is the functor $(g_1')_*$ applied to the following diagram evaluated at $((g_2')^*E, K)$:

$$\begin{tikzcd}
(t')^*(t'_*(-)\otimes -) \arrow[r, "\mathrm{PF}"] \arrow[d, "M_{(-)^*}"'] & (t')^*t'_*(-\otimes (t')^*(-)) \arrow[d, "\xi'"] \\
(t')^*t'_*(-)\otimes (t')^*(-) \arrow[r, "\xi'"'] & (-)\otimes (t')^*(-)
\end{tikzcd}$$

which commutes by definition of the projection formula.

Now, for (4.9). Note that it is the following diagram evaluated at $(g_2')^*E\otimes (t')^*K$:

$$\begin{tikzcd}
t_2^*(f_1')_*t'_* \arrow[r, "F_{(-)_*}"] \arrow[d, "\mathrm{BC}_2"'] & t_2^*(t_2)_*(g_1')_* \arrow[d, "\xi_2"] \\
(g_1')_*(t')^*t'_* \arrow[r, "\xi'"'] & (g_1')_*
\end{tikzcd}$$

which commutes by definition of the base change formula.

Lastly, we check (4.10). However, it is the functor $(g_1')_*(-\otimes (t')^*K)$ applied to the following diagram evaluated at E:

$$\begin{tikzcd}
(t')^*t'_*(g_2')^* \arrow[r, "\xi'"] & (g_2')^* \\
(t')^*(f_2')^*(t_1)_* \arrow[r, "F_{(-)^*}"'] \arrow[u, "\mathrm{BC}_1"] & (g_2')^*t_1^*(t_1)_* \arrow[u, "\xi_1"']
\end{tikzcd}$$

which commutes by definition of the base change formula. □

Remark 4.24. Assume the hypothesis of Lemma 4.23. Denote by ζ^i the unit of $t_i^* \dashv (t_i)_*$ if $i\in\{1,2\}$. From [GLMP25, Lemma 4.1] and Lemma 4.23, we have that the following diagram is commutative:

$$\begin{tikzcd}
\Phi_K \arrow[r, "\zeta^2_{\Phi_K}"] \arrow[d, "\Phi_K(\zeta^1)"'] & (t_2)_*t_2^*\Phi_K \arrow[d, "(t_2)_*(\beta^K)"] \\
\Phi_K(t_1)_*t_1^* \arrow[r, "\alpha^K_{t_1^*}"] & (t_2)_*\Phi_{(t')^*K}t_1^*.
\end{tikzcd} \tag{4.11}$$

Proposition 4.25. *Consider Setup 4.1 with t affine. Let $K\in D_{\mathrm{qc}}(Y_1\times_S Y_2)$ be pseudocoherent and relatively perfect over Y_2. Denote by K' the kernel of the integral transform obtained in Proposition 4.17 which is right adjoint to Φ_K on D_{qc}. Then, after possibly replacing $\Phi_{\mathbf{L}(t')^*K}$ by an isomorphic functor, we have a morphism of adjoint pairs:*

$$\begin{tikzcd}
D_{\mathrm{qc}}(Y_1) \arrow[r, "\mathbf{L}t_1^*"] \arrow[d, shift right, "\Phi_K"'] \arrow[d, phantom, "\dashv"] & D_{\mathrm{qc}}(Y_1\times_S T) \arrow[d, shift right, "\Phi_{\mathbf{L}(t')^*K}"'] \arrow[d, phantom, "\dashv"] \\
D_{\mathrm{qc}}(Y_2) \arrow[r, "\mathbf{L}t_2^*"'] \arrow[u, shift right, "\Phi_{K'}"'] & D_{\mathrm{qc}}(Y_2\times_S T) \arrow[u, shift right, "\Phi_{\mathbf{L}(t')^*K'}"']
\end{tikzcd}$$

with left and right comparison transformations given respectively by the isomorphisms

$$(\beta^K)^{-1}\colon \Phi_{\mathbf{L}(t')^*K} \circ \mathbf{L}t_1^* \to \mathbf{L}t_2^* \circ \Phi_K,$$
$$\beta^{K'}\colon \mathbf{L}t_1^* \circ \Phi_{K'} \to \Phi_{\mathbf{L}(t')^*K'} \circ \mathbf{L}t_2^*.$$

Proof. Write η (resp. η') for the unit of the adjunction $\Phi_K \dashv \Phi_{K'}$ (resp. of $\Phi_{(t')^*K} \dashv \Phi_{(t')^*K'}$). By [GLMP25, Lemma 4.1(2)], proving the Proposition amounts to showing the following diagram commutes:

$$\begin{tikzcd}
\mathbf{L}t_1^* \arrow[r, "\eta'_{\mathbf{L}t_1^*}"] \arrow[d, "\mathbf{L}t_1^*(\eta)"'] & \Phi_{\mathbf{L}(t')^*K'}\Phi_{\mathbf{L}(t')^*K}\mathbf{L}t_1^* \\
\mathbf{L}t_1^*\Phi_{K'}\Phi_K \arrow[r, "\beta^{K'}_{\Phi_K}"'] & \Phi_{\mathbf{L}(t')^*K'}\mathbf{L}t_2^*\Phi_K. \arrow[u, "\Phi_{\mathbf{L}(t')^*K'}(\beta^K)"']
\end{tikzcd} \tag{4.12}$$

This can be argued verbatim to [GLMP25, Proposition 4.7]. In particular, Proposition 4.17 and Lemma 4.23 are the key steps that allow one to adapt the proof of loc. cit. We justify the claim below.

As in the proof of Lemma 4.23, we omit $\mathbf{L}$'s and $\mathbf{R}$'s in the notation for the derived functors to ease notation. Now, the bad news is that (4.12) needs not commute. Although, the good news is that it does commute if we replace $\Phi_{(t')^*K}$ by some appropriate isomorphic functor. Specifically, we claim there is an adjunction $\widetilde{\Phi}_{(t')^*K} \dashv \Phi_{(t')^*K'}$ with unit $\tilde{\eta}'$ making (4.12) commute (i.e. inscribing $\widetilde{(-)}$ to $\Phi_{(t')^*K}$ and η') and that moreover there is an isomorphism $\theta\colon \Phi_{(t')^*K} \cong \widetilde{\Phi}_{(t')^*K}$ satisfying

$$\Phi_{(t')^*K'}(\theta) \circ \eta' = \tilde{\eta}'. \tag{4.13}$$

Assume that for every $E \in D_{\mathrm{qc}}(Y_1)$ we have shown:

($\mathscr{P}(E)$) The object $\Phi_{(t')^*K}t_1^*E$ with the morphism $\Phi_{(t')^*K}(\beta^K_E) \circ \beta^{K'}_{\Phi_K E} \circ t_1^*(\eta_E)$ is a reflection of t_1^*E along $\Phi_{(t')^*K'}$ (in the sense of [Bor94, Definition 3.1.1]).

Then we can define $\widetilde{\Phi}_{(t')^*K}$ by means of [Bor94, Proposition 3.1.3]. On the one hand, we have $\Phi_{(t')^*K} \dashv \Phi_{(t')^*K'}$ with unit η'; thus by [Bor94, Theorem 3.1.5] the pair $(\Phi_{(t')^*K}A, \eta'_A)$ is a reflection of $A \in D_{\mathrm{qc}}(Y_1 \times_S T)$ along $\Phi_{(t')^*K'}$. Now define $\tilde{\eta}' = \{\tilde{\eta}'_A\}_{A \in D_{\mathrm{qc}}(Y_1 \times_S T)}$ by setting $\tilde{\eta}'_A := \eta'_A$ if A is not in the image of $t_1^*\colon D_{\mathrm{qc}}(Y_1) \to D_{\mathrm{qc}}(Y_1 \times_S T)$ and setting $\tilde{\eta}'_{t_1^*E}$, $E \in D_{\mathrm{qc}}(Y_1)$, to be the unique top morphism in (4.12) making it commute. By [Bor94, Proposition 3.1.3], there is a unique functor $\widetilde{\Phi}_{(t')^*K}\colon D_{\mathrm{qc}}(Y_1 \times_S T) \to D_{\mathrm{qc}}(Y_2 \times_S T)$ whose action on objects coincides with $\Phi_{(t')^*K}$ and such that $\tilde{\eta}'\colon \mathrm{id} \to \widetilde{\Phi}_{(t')^*K'}\Phi_{(t')^*K}$ is a natural transformation. Moreover, by [Bor94, Theorem 3.1.5], we have $\widetilde{\Phi}_{(t')^*K} \dashv \Phi_{(t')^*K'}$ with unit $\tilde{\eta}'$. Finally, by [Rie17, Proposition 4.4.1], there is a natural isomorphism $\theta\colon \Phi_{(t')^*K} \cong \widetilde{\Phi}_{(t')^*K}$ such that (4.13) holds.

It is left for us to show $\mathscr{P}(E)$. So, let $A \in D_{\mathrm{qc}}(Y_2 \times_S T)$ and consider some $\gamma\colon t_1^*E \to \Phi_{(t')^*K'}A$. Our goal is to check that there is a unique morphism $\tilde{\gamma}\colon \Phi_{(t')^*K}t_1^*E \to A$ such that the diagram

$$
\begin{tikzcd}
t_1^*E \arrow[d, "t_1^*(\eta_E)"'] \arrow[dr, "\gamma"] & \\
t_1^*\Phi_{K'}\Phi_K E \arrow[d, "\beta^{K'}_{\Phi_K(E)}"'] & \Phi_{(t')^*K'}A \\
\Phi_{(t')^*K'}t_2^*\Phi_K E \arrow[d, "\Phi_{(t')^*K'}(\beta^K_E)"'] & \\
\Phi_{(t')^*K'}\Phi_{(t')^*K}t_1^*E \arrow[uur, "\Phi_{(t')^*K'}(\tilde{\gamma})"'] &
\end{tikzcd}
\tag{4.14}
$$

commutes. Let $i \in \{1, 2\}$. Denote by ζ^i the unit of $t_i^* \dashv (t_i)_*$. Given morphisms

$$\phi\colon t_i^*M \to B, \quad \psi\colon C \to (t_i)_*D, \quad \rho\colon L \to \Phi_{K'}P,$$

we will use notations

$$\phi^{\flat_i}\colon M \to (t_i)_*B, \quad \psi^{\sharp_i}\colon t_i^*C \to D, \quad \rho^{\sharp}\colon \Phi_K L \to P$$

for the adjunct morphisms of ϕ, ψ, ρ with respect to the adjunctions $t_i^* \dashv (t_i)_*$ and $\Phi_K \dashv \Phi_{K'}$. Suppose there is $\tilde{\gamma}$ turning (4.14) into a commutative diagram. Then $\gamma^{\flat_1}\colon E \to (t_1)_*\Phi_{(t')^*K'}A$ equals the outer clockwise composite in the following diagram:

$$
\begin{tikzcd}
E \arrow[r, "\zeta^1_E"] \arrow[d, "\eta_E"'] \arrow[dddd, bend right=60, "\delta"'] & (t_1)_*t_1^*E \arrow[d, "(t_1)_*t_1^*(\eta_E)"] \\
\Phi_{K'}\Phi_K E \arrow[r, "\zeta^1_{\Phi_{K'}\Phi_K E}"] \arrow[d, "\Phi_{K'}(\zeta^2_{\Phi_K E})"'] & (t_1)_*t_1^*\Phi_{K'}\Phi_K E \arrow[d, "(t_1)_*(\beta^{K'}_{\Phi_K(E)})"] \\
\Phi_{K'}(t_2)_*t_2^*\Phi_K E \arrow[r, "\alpha^{K'}_{t_2^*\Phi_K E}"] \arrow[d, "\Phi_{K'}(t_2)_*(\beta^K_E)"] & (t_1)_*\Phi_{(t')^*K'}t_2^*\Phi_K E \arrow[d, "(t_1)_*\Phi_{(t')^*K'}(\beta^K_E)"] \\
\Phi_{K'}(t_2)_*\Phi_{(t')^*K}t_1^*E \arrow[d, "\Phi_{K'}(t_2)_*(\tilde{\gamma})"] & (t_1)_*\Phi_{(t')^*K'}\Phi_{(t')^*K}t_1^*E \arrow[d, "(t_1)_*\Phi_{(t')^*K'}(\tilde{\gamma})"] \\
\Phi_{K'}(t_2)_*A \arrow[r, "\alpha^{K'}_A"'] & (t_1)_*\Phi_{(t')^*K'}A.
\end{tikzcd}
$$

In this diagram, we note the upper square commutes by naturality of ζ^1, the middle square commutes by Lemma 4.23 (it is (4.11) with K replaced by K'), and the bottom rectangle commutes by naturality of $\alpha^{K'}$. Set δ for the left vertical composite in the last diagram, i.e. $\delta = (\alpha^{K'}_A)^{-1} \circ \gamma^{\flat_1}\colon E \to \Phi_{K'}(t_2)_*A$. Then it follows that

$$\delta^{\sharp}\colon \Phi_K E \xrightarrow{\zeta^2_{\Phi_K E}} (t_2)_*t_2^*\Phi_K E \xrightarrow{(t_2)_*(\beta^K_E)} (t_2)_*\Phi_{(t')^*K}t_1^*E \xrightarrow{(t_2)_*(\tilde{\gamma})} (t_2)_*A,$$

which tells us

$$(\delta^{\sharp})^{\sharp_2}\colon t_2^*\Phi_K E \xrightarrow{\beta^K_E} \Phi_{(t')^*K}t_1^*E \xrightarrow{\tilde{\gamma}} A.$$

Consequently, we have

$$\begin{aligned}\tilde{\gamma} &= (\delta^{\sharp})^{\sharp_2} \circ (\beta_E^K)^{-1} \\ &= ([(\alpha_A^{K'})^{-1} \circ \gamma^{\flat_1}]^{\sharp})^{\sharp_2} \circ (\beta_E^K)^{-1},\end{aligned}$$

which shows uniqueness of $\tilde{\gamma}$. Conversely, defining $\tilde{\gamma}$ by the last formula and reading the previous derivations in reverse order, it follows that (4.14) commutes. □

Remark 4.26. Consider the situation as in Proposition 4.25. Denote ε and ε' respectively for the counits of the adjunctions $\Phi_K \dashv \Phi_{K'}$ and $\Phi_{\mathbf{L}(t')^*K} \dashv \Phi_{\mathbf{L}(t')^*K'}$. From [GLMP25, Lemma 4.1], it follows that (after possibly replacing $\Phi_{\mathbf{L}(t')^*K}$ by an isomorphic functor) the following diagram commutes:

$$\begin{tikzcd} \Phi_{\mathbf{L}(t')^*K}\mathbf{L}t_1^*\Phi_{K'} \arrow[d, "\Phi_{\mathbf{L}(t')^*K}(\beta^{K'})"'] & \mathbf{L}t_2^*\Phi_K\Phi_{K'} \arrow[l, "\beta^K_{\Phi_{K'}}"'] \arrow[d, "\mathbf{L}t_2^*(\varepsilon)"] \\ \Phi_{\mathbf{L}(t')^*K}\Phi_{\mathbf{L}(t')^*K'}\mathbf{L}t_2^* \arrow[r, "\varepsilon'_{\mathbf{L}t_2^*}"] & \mathbf{L}t_2^*. \end{tikzcd} \tag{4.15}$$

Remark 4.27. On a locally Noetherian algebraic space, any closed point can be represented by a morphism of finite type. See [Sta26, 0H1U & 06QX].

Lemma 4.28. *Let $F\colon \mathcal{T} \leftrightarrows \mathcal{S}\colon G$ be a pair of exact adjoint functors between compactly generated triangulated categories. Assume G preserves small coproducts. Then F is an equivalence if, and only if, F restricts to an equivalence $\mathcal{T}^c \to \mathcal{S}^c$. A similar statement holds for F being fully faithful. Also, if F restricts to a fully faithful functor $\mathcal{T}^c \to \mathcal{S}^c$ and $F(\mathcal{T}^c)$ compactly generates $\mathcal{S}$, then $F\colon \mathcal{T} \to \mathcal{S}$ is an equivalence. Additionally, if F restricts to a fully faithful functor $\mathcal{T}^c \to \mathcal{S}^c$, then F is fully faithful $\mathcal{T} \to \mathcal{S}$.*

Proof. It is straightforward to check that F being an equivalence induces an equivalence $\mathcal{T}^c \to \mathcal{S}^c$. We check the converse. To do so, we show that F and G are both fully faithful. Denote the counit and unit of the adjunction respectively by ϵ and η. Let $\mathcal{T}'$ and $\mathcal{S}'$ be the strictly full subcategories respectively of $\mathcal{T}$ and $\mathcal{S}$ consisting of objects where η and ϵ are isomorphisms. One may verify that $\mathcal{T}'$ and $\mathcal{S}'$ are closed under shifts, iterated cones, and small coproducts (hence, homotopy colimits). However, [Sta26, Tag 09SN] tells us each object in $\mathcal{T}$ and $\mathcal{S}$ can be obtained respectively by $\mathcal{T}^c$ and $\mathcal{S}^c$ using these operations. Hence, $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{S} \subseteq \mathcal{S}'$, which completes the proof of the first claim.

The remaining claims follow similarly. Indeed, we show the very last claim. Let $\mathcal{T}'$ denote the strictly full subcategory of $E \in \mathcal{T}$ such that $\eta_E\colon E \to GF(E)$ is an isomorphism. It can be checked that $\mathcal{T}'$ is a triangulated subcategory of $\mathcal{T}$ which is closed under small coproducts. By hypothesis, it follows that $\mathcal{T}^c \subseteq \mathcal{T}'$. Since $\mathcal{T}$ is compactly generated, it follows that $\mathcal{T} = \mathcal{T}'$. Therefore, η_E is an isomorphism for all $E \in \mathcal{T}$. □

Lemma 4.29. *Let $f_1\colon Y_1 \to S$ and $f_2\colon Y_2 \to S$ be proper flat morphisms of Noetherian algebraic spaces. Suppose $K \in D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$ is relatively perfect over each Y_i. Denote by K' the kernel of the integral transform obtained in Proposition 4.17 which is right adjoint to Φ_K on D_{qc}. Then the following are equivalent:*

(1) *Φ_K is fully faithful on D_{qc}*
(2) *Φ_K restricts to a fully faithful functor on D^b_{coh}*
(3) *Φ_K restricts to a fully faithful functor* Perf.

Furthermore, if K' is relatively perfect over Y_1, then a similar statement holds for Φ_K and $\Phi_{K'}$ forming an adjoint equivalence.

Proof. We first prove fully faithfulness. By Lemma 4.28, it follows that (1) $\iff$ (3). As Φ_K restricts to a functor on D^b_{coh}, (1) $\implies$ (2). Since K is relatively perfect over both Y_i, it also restricts to a functor on Perf, and so (2) $\implies$ (3).

Now we check the case of Φ_K and $\Phi_{K'}$ forming an adjoint equivalence. By Corollary 4.18, Φ_K and $\Phi_{K'}$ restrict to give an adjoint pair on D^b_{coh}. In fact, Proposition 4.17 says K' is bounded and pseudocoherent. Applying Corollary 3.45, K' is relatively perfect over Y_2. Moreover, from the hypotheses that K' is relatively perfect over Y_1, Corollary 3.45 implies $\Phi_{K'}$ restricted to $\mathrm{Perf}(Y_2)$ factors through $\mathrm{Perf}(Y_1)$. To check Φ_K and $\Phi_{K'}$ form an adjoint equivalence, it suffices to check they are both fully faithful. Therefore, we can apply the first case above. □

Proposition 4.30. *Consider Setup 4.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over each Y_i. If $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful on D^b_{coh} where t is one of the following:*

(1) t is affine and faithfully flat
(2) for every $t\colon \mathrm{Spec}(k) \to S$ of finite type which represents any closed point $p \in S$

then so is Φ_K. Furthermore, if K' is relatively perfect over Y_1, then a similar statement holds for Φ_K and $\Phi_{K'}$ forming an adjoint equivalence.

Proof. We give an alternate proof to [GLMP25, Proposition 4.9]. We prove the claim for full faithfulness. We first recall some properties satisfied by any morphism t appearing in the statement. By Theorem 4.9, $\mathbf{L}(t')^*K$ is relatively perfect over each $Y_i \times_S T$. By Proposition 4.17, Φ_K and $\Phi_{K'}$, as well as $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$, form adjoint pairs on D_{qc}. Denote by η (resp. η') the unit of the adjunction Φ_K and $\Phi_{K'}$ (resp. $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$) on D_{qc}. By Corollary 4.18, these adjoint pairs restrict to D^b_{coh}. Moreover, Corollary 3.45 implies that Φ_K and $\Phi_{\mathbf{L}(t')^*K}$ restrict to functors on Perf.

We start by proving (1). Let $E \in D^b_{\mathrm{coh}}(Y_1)$. Since t is faithfully flat, t_1 is as well, and so $\mathbf{L}t_1^*E \in D^b_{\mathrm{coh}}(Y_1 \times_S T)$. By (4.12), it follows that

$$\mathrm{cone}(\eta'_{\mathbf{L}t_1^*E}) \cong \mathbf{L}t_1^* \, \mathrm{cone}(\eta_E).$$

However, $\Phi_{\mathbf{L}(t')^*K}$ restricts to a fully faithful functor on D^b_{coh}, and so $\mathrm{cone}(\eta'_{\mathbf{L}t_1^*E}) \cong 0$. Then Lemma 2.9 implies that $\mathrm{cone}(\eta_E) \cong 0$. Hence, Φ_K restricts to a fully faithful functor on D^b_{coh} because E was arbitrary.

Lastly we check (2). By Lemma 4.29, it suffices to check Φ_K restricts to a fully faithful functor on $D^b_{\mathrm{coh}}(Y_1)$. Choose any $P \in D^b_{\mathrm{coh}}(Y_1)$. Since Φ_K restricts to a functor on D^b_{coh}, we know that $\mathrm{cone}(\eta_P) \in D^b_{\mathrm{coh}}(Y_1)$. Let $p \in |Y_1|$ be a closed point. Suppose $s\colon \mathrm{Spec}(k) \to Y_1$ represents p and is of finite type. Since f_1 is proper, Lemma 2.4 says $t := f_1 \circ s$ represents the closed point $f_1(p) \in |S|$ and is of finite type.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[dr, "h"] \arrow[drr, bend left, "1_{\mathrm{Spec}(k)}"] \arrow[ddr, bend right, "s"'] & & \\
& Y_1 \times_S \mathrm{Spec}(k) \arrow[r, "f_1'"] \arrow[d, "t_1"] & \mathrm{Spec}(k) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"'] & S.
\end{tikzcd}$$

By (4.12), we obtain that

$$\mathrm{cone}(\eta'_{\mathbf{L}t_1^* P}) \cong \mathbf{L}t_1^* \,\mathrm{cone}(\eta_P).$$

By hypothesis, $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful, and so $\mathrm{cone}(\eta'_{\mathbf{L}t_1^* P}) \cong 0$ (see Lemma 4.29). Then

$$\mathbf{L}s^* \,\mathrm{cone}(\eta_P) \cong \mathbf{L}h^* \mathbf{L}t_1^* \,\mathrm{cone}(\eta_P) \cong 0.$$

By Lemma 2.5, we see that $p \notin \mathrm{supp}(\mathrm{cone}(\eta_P))$. Since p was an arbitrary closed point of Y_1, it follows that $\mathrm{supp}(\mathrm{cone}(\eta_P)) = \emptyset$. In other words, $\mathrm{cone}(\eta_P)$ is the zero object, which completes the proof for fully faithfulness.

To prove the last claim for adjoint equivalence, the same argument above can be adapted with the counit of the adjoint pair. This completes the proof. □

Proposition 4.31. *Consider Setup 4.1 where t is affine. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over Y_2. Denote by K' the kernel of the integral transform obtained in Proposition 4.17 which is right adjoint to Φ_K on D_{qc}. Then, after possibly replacing $\Phi_{K'}$ by an isomorphic functor, we have a morphism of adjoint pairs as depicted in the diagram*

$$\begin{tikzcd}
D_{\mathrm{qc}}(Y_1 \times_S T) \arrow[r, "\mathbf{R}(t_1)_*"] \arrow[d, shift right, "\Phi_{\mathbf{L}(t')^*K}"'] & D_{\mathrm{qc}}(Y_1) \arrow[d, shift right, "\Phi_K"'] \\
D_{\mathrm{qc}}(Y_2 \times_S T) \arrow[r, "\mathbf{R}(t_2)_*"'] \arrow[u, shift right, "\Phi_{\mathbf{L}(t')^*K'}"'] \arrow[u, phantom, "\dashv"] & D_{\mathrm{qc}}(Y_2) \arrow[u, shift right, "\Phi_{K'}"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

with left and right comparison transformations given respectively by the isomorphisms

$$\alpha^K \colon \Phi_K \circ \mathbf{R}(t_1)_* \to \mathbf{R}(t_2)_* \circ \Phi_{\mathbf{L}(t')^*K},$$
$$(\alpha^{K'})^{-1} \colon \mathbf{R}(t_1)_* \circ \Phi_{\mathbf{L}(t')^*K'} \to \Phi_{K'} \circ \mathbf{R}(t_2)^*.$$

Proof. This argument is dual to that of Proposition 4.25. □

Remark 4.32. Consider the situation as in Proposition 4.31. Denote ε and ε' respectively for the counits of the adjunctions $\Phi_K \dashv \Phi_{K'}$ and $\Phi_{\mathbf{L}(t')^*K} \dashv \Phi_{\mathbf{L}(t')^*K'}$. From [GLMP25, Lemma 4.1], it follows that (after possibly replacing $\Phi_{K'}$ by an isomorphic functor) the following diagrams commute:

$$\begin{tikzcd}
\Phi_K \mathbf{R}(t_1)_* \Phi_{\mathbf{L}(t')^*K'} \arrow[r, "\alpha^K_{\Phi_{\mathbf{L}(t')^*K'}}"] & \mathbf{R}(t_2)_* \Phi_{\mathbf{L}(t')^*K} \Phi_{\mathbf{L}(t')^*K'} \arrow[d, "\mathbf{R}(t_2)_*(\varepsilon')"] \\
\Phi_K \Phi_{K'} \mathbf{R}(t_2)_* \arrow[u, "\Phi_K(\alpha^{K'})"] \arrow[r, "\varepsilon_{\mathbf{R}(t_2)_*}"'] & \mathbf{R}(t_2)_*
\end{tikzcd} \tag{4.16}$$

$$\begin{tikzcd} \mathbf{R}(t_1)_* \arrow[r, "\eta_{\mathbf{R}(t_1)_*}"] \arrow[d, "\mathbf{R}(t_1)_*(\eta')"'] & \Phi_{K'}\Phi_K\mathbf{R}(t_1)_* \arrow[d, "\Phi_{K'}(\alpha^K)"] \\ \mathbf{R}(t_1)_*\Phi_{\mathbf{L}(t')^*K'}\Phi_{\mathbf{L}(t')^*K} & \Phi_{K'}\mathbf{R}(t_2)_*\Phi_{\mathbf{L}(t')^*K}. \arrow[l, "\alpha^{K'}_{\Phi_{\mathbf{L}(t')^*K}}"] \end{tikzcd} \tag{4.17}$$

Proposition 4.33. *Consider Setup 4.1 where t is affine. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over each Y_i. If Φ_K is fully faithful on D^b_{coh}, then so is $\Phi_{\mathbf{L}(t')^*K}$. Furthermore, if K' is relatively perfect over Y_1, then a similar statement holds for Φ_K and $\Phi_{K'}$ forming an adjoint equivalence.*

Proof. We start by proving the case of fully faithfulness. By Theorem 4.9, $\mathbf{L}(t')^*K$ is relatively perfect over each $Y_i \times_S T$. By Proposition 4.17, Φ_K and $\Phi_{K'}$, as well as $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$, form adjoint pairs on D_{qc}. Denote by η (resp. η') the unit of the adjoint pair Φ_K and $\Phi_{K'}$ (resp. $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$) on D_{qc}. By Corollary 4.18, these adjoint pairs restrict to D^b_{coh}. Moreover, Corollary 3.45 implies that Φ_K and $\Phi_{\mathbf{L}(t')^*K}$ restrict to functors on Perf. Let $G \in D_{\mathrm{qc}}(Y_1)$ be a compact generator for $D_{\mathrm{qc}}(Y_1)$. By Lemma B.3, $\mathbf{L}t_1^*G$ is a compact generator for $D_{\mathrm{qc}}(Y_1 \times_S T)$. By (4.12), we obtain

$$\operatorname{cone}(\eta'_{\mathbf{L}t_1^*G}) \cong \mathbf{L}t_1^* \operatorname{cone}(\eta_G).$$

Since Φ_K is fully faithful on Perf, we have that $\operatorname{cone}(\eta'_{\mathbf{L}t_1^*G}) \cong 0$ because $\operatorname{cone}(\eta_G) \cong 0$. Let $\mathcal{A}$ be the collection of objects E in $D_{\mathrm{qc}}(Y_1 \times_S T)$ such that η'_E is an isomorphism. We have shown that $\mathbf{L}t_1^*G \in \mathcal{A}$. Therefore, Lemma 4.28 implies $\mathcal{A} = D_{\mathrm{qc}}(Y_1 \times_S T)$, which completes the proof for fully faithfulness. To check the last claim, the same argument above can be adapted with the counit of the adjoint pair. This completes the proof. □

Theorem 4.34. *Consider Setup 4.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over each Y_i. Then the following are equivalent:*

(1) Φ_K is fully faithful on D^b_{coh}
(2) $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful on D^b_{coh} for any affine morphism t
(3) $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful on D^b_{coh} for every $t\colon \operatorname{Spec}(k) \to S$ of finite type that represents any closed point $p \in |S|$
(4) for any closed point $p \in |S|$ there exists a representative $t\colon \operatorname{Spec}(k) \to S$ of p such that $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful on D^b_{coh}.

Furthermore, if K' is relatively perfect over Y_1, then a similar statement holds for Φ_K and $\Phi_{K'}$ forming an adjoint equivalence (in fact, one can detect equivalences on Perf *or even D_{qc}).*

Proof. (1) $\implies$ (2) follows from Proposition 4.33; it is straightforward to check that (2) $\implies$ (3) using [Sta26, Tag 09TF]; (3) $\implies$ (1) follows from Proposition 4.30. Clearly, (3) $\implies$ (4), and so we check the converse. Choose a closed point $p \in |S|$. Assume there exists a representative $t\colon \operatorname{Spec}(k) \to S$ of p such that $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful on D^b_{coh}. Pick any $h\colon \operatorname{Spec}(k') \to S$ of finite type that represents p. There exist a field ℓ and a commutative diagram,

$$\begin{tikzcd} \operatorname{Spec}(\ell) \arrow[r, "a"] \arrow[d, "b"'] & \operatorname{Spec}(k') \arrow[d, "h"] \\ \operatorname{Spec}(k) \arrow[r, "t"'] & S. \end{tikzcd}$$

Note that a, b are affine and faithfully flat. The hypothesis is that fully faithfulness occurs after base change along t. We can apply Proposition 4.33 to realize fully faithfulness occurs after base change along $t \circ b$. Since b is affine and faithfully flat, Proposition 4.30 implies fully faithfulness occurs after base change along h (e.g. use that $h \circ a = t \circ b$). Lastly, applying Lemma 4.29, a similar argument holds for equivalences. □

5. Applications

5.1. Elliptic fibrations.

Lemma 5.1. *Let $f : Y \to X$ be a concentrated proper flat morphism of Noetherian algebraic spaces. Denote by C_f the cone of the unit $\mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ where $\Delta_f : Y \to Y \times_X Y$ is the diagonal. Set $p, q : Y \times_X Y \to Y$ the natural morphisms. If f has affine diagonal, then C_f is relatively perfect for p and q.*

Proof. We prove the case for q because the other is similar (e.g. switch notation). There exists a fibered square

$$\begin{tikzcd} Y \times_X Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Consider the distinguished triangle

$$C_f[-1] \to \mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \to C_f.$$

Since f has affine and proper diagonal, the natural morphism $(\Delta_f)_*\mathcal{O}_Y \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is an isomorphism. Now, if we extend to the long exact sequence in cohomology with the distinguished triangle above, we see that $C_f[-1]$ is a coherent sheaf on $Y \times_X Y$ because $\mathcal{O}_{Y\times_X Y} \to (\Delta_f)_*\mathcal{O}_Y$ is surjective since Δ_f is a closed immersion. Let $E \in D^b_{\mathrm{qc}}(Y)$. Since q is flat, we know that $\mathbf{L}q^*E \in D^b_{\mathrm{qc}}(Y \times_X Y)$. Consider the distinguished triangle obtained by tensoring with $\mathbf{L}q^*E$,

$$C_f[-1] \otimes^{\mathbf{L}} \mathbf{L}q * E \to \mathcal{O}_{Y\times_X Y} \otimes^{\mathbf{L}} \mathbf{L}q^*E \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}q^*E \to C_f \otimes^{\mathbf{L}} \mathbf{L}q^*E.$$

If we can show that $\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}q^*E \in D^b_{\mathrm{qc}}(Y \times_X Y)$, then we are done. Now, by projection formula, we have

$$\begin{aligned} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}q^*E &\cong \mathbf{R}(\Delta_f)_*\mathbf{L}\Delta_f^*\mathbf{L}q^*E \\ &\cong \mathbf{R}(\Delta_f)_*\mathbf{L}(q \circ \Delta_f)^*E \\ &\cong \mathbf{R}(\Delta_f)_*E \qquad\qquad (q \circ \Delta_f = 1_Y). \end{aligned}$$

Since Δ_f is affine, it follows that $\mathbf{R}(\Delta_f)_*$ is t-exact and conservative. Hence, $\mathbf{R}(\Delta_f)_*E \in D^b_{\mathrm{qc}}(Y \times_X Y)$, which completes the proof. □

Reminder 5.2. Recall that an **elliptic X-fibration** is a proper Gorenstein morphism $Y \to X$ between Noetherian algebraic spaces such that for each $p \in |X|$ there is a representative $\mathrm{Spec}(k) \to X$ of p such that $Y \times_X \mathrm{Spec}(k)$ is a geometrically integral 1-dimensional k-scheme of genus[1].

[1]We follow [Sta26, Tag 0BY7] for the definition of 'genus'.

Example 5.3. Let k be a field. Consider a geometrically integral 1-dimensional k-scheme Y of genus one with trivial dualizing sheaf. Then the natural morphism $Y \times_k Y \to Y$ is an elliptic fibration.

To see, consider a morphism $\operatorname{Spec}(L) \to Y$ from a field. Note that the base change of $Y \times_k Y$ along $\operatorname{Spec}(L) \to Y$ is the base change Y along L/k. As $Y \times_k \operatorname{Spec}(L)$ is geometrically integral (and hence, connected), [Sta26, Tag 0FD2] says that

$$H^0(Y \times_k \operatorname{Spec}(L), \mathcal{O}_{Y \times_k \operatorname{Spec}(L)}) = L.$$

From base change, we see that $Y \times_k \operatorname{Spec}(L)$ is proper and Gorenstein over L. Hence, by [Sta26, Tag 0A26], $Y \times_k \operatorname{Spec}(L)$ is projective over L. Moreover, [Sta26, Tag 0BY9] tells us that $Y \times_k \operatorname{Spec}(L)$ has genus one. Applying [Sta26, Tag 0FW1], it follows that $Y \times_k \operatorname{Spec}(L)$ has trivial dualizing sheaf. Consequently, the base change of $Y \times_k Y$ over $\operatorname{Spec}(L) \to Y$ is projective, geometrically integral, of Krull dimension one, genus one, and with trivial dualizing sheaf.

More generally, if $Y \to S$ is an elliptic fibration, then $Y \times_S Y \to Y$ is an elliptic fibration. This can be argued in a similar fashion to the case $S = \operatorname{Spec}(k)$ for a field k above. In fact, if $Y \to S$ and $X \to S$ are elliptic fibrations, then the natural projections of $X \times_S Y$ to X and Y are elliptic fibrations. Again, the argument follows analogously to that above.

Proposition 5.4. *Let X be a Noetherian algebraic space. Let $T \to X$ be an affine morphism from a Noetherian algebraic space and $f \colon Y \to X$ a proper flat morphism from a Noetherian algebraic space. Denote by $\mathcal{I}_{\Delta_f}$ (resp. $\mathcal{I}_{\Delta'_f}$) the ideal sheaf of the diagonal morphism $\Delta_f \colon Y \to Y \times_X Y$ (resp. $\Delta_{f'} \colon Y \times_X T \to Y \times_X Y \times_X T$). Then $\mathbf{L}(t')^* \mathcal{I}_{\Delta_f} \cong \mathcal{I}_{\Delta'_f}$ where $t' \colon Y \times_X Y \times_X T \to Y \times_X Y$ is the natural morphism.*

Proof. Since f is proper, the diagonal Δ_f is a closed immersion. Consider the commutative diagram obtained by base changes,

$$\begin{tikzcd}
Y \times_X Y \times_X T \arrow[rr, "q'"] \arrow[dr, "p'"] \arrow[dd, "t'"'] & & Y \times_X T \arrow[dr, "f'"] \arrow[dd, dashed, "s" near end] & \\
& Y \times_X T \arrow[rr, "f'" near start] \arrow[dd, "s" near end] & & T \arrow[dd, "t"] \\
Y \times_X Y \arrow[rr, dashed, "q" near start] \arrow[dr, "p"'] & & Y \arrow[dr, "f"] & \\
& Y \arrow[rr, "f"'] & & X.
\end{tikzcd}$$

Note that base change shows that p, q are flat. We obtain a diagram of fibered squares

$$\begin{tikzcd}
Y \times_X T \arrow[r, "\Delta_{f'}"] \arrow[d, "s"'] & Y \times_X Y \times_X T \arrow[r, "p'"] \arrow[d, "t'"'] & Y \times_X T \arrow[r, "f'"] \arrow[d, "s"] & T \arrow[d, "t"] \\
Y \arrow[r, "\Delta_f"'] & Y \times_X Y \arrow[r, "p"'] & Y \arrow[r, "f"'] & X.
\end{tikzcd}$$

The leftmost square occurs from the fact that $\Delta_{f'}$ is the base change of Δ_f along t' (see e.g. proof of [Sta26, Tag 03KL]). Since t is affine, base change tells us both s and t' are affine, and hence representable by schemes. By [Sta26, Tag 07U8], there is an isomorphism

$$\begin{aligned}\mathbf{R}(\Delta_{f'})_*\mathbf{L}s^*\mathcal{O}_Y &\cong \mathbf{R}(\Delta_{f'})_*s^*\mathcal{O}_Y\\ &\cong \mathbf{R}(\Delta_{f'})_*\mathcal{O}_{Y\times_X T}\\ &\cong (t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y\\ &\cong (t')^*(\Delta_f)_*\mathcal{O}_Y\end{aligned}$$

where $(t')^*$ is underived.

We want to show that the natural morphism $(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \to \mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is an isomorphism. This amounts to checking that $\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is concentrated in degree zero. Since $\mathbf{R}t'_*$ is t-exact and conservative, it suffices to check that $\mathbf{R}t'_*\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is concentrated in degree zero. Indeed, t-exactness gives

$$\mathcal{H}^i(\mathbf{R}t'_*\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y) \cong t'_*\mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y),$$

whereas conservativeness shows

$$t'_*\mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y) \cong 0 \implies \mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y) \cong 0.$$

By projection formula, we have

$$\mathbf{R}t'_*\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \cong \mathbf{R}t'_*\mathcal{O}_{Y\times_X Y\times_X T} \otimes^{\mathbf{L}} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y.$$

Note that

$$\mathcal{O}_{Y\times_X Y\times_X T} \cong \mathbf{L}(f'\circ p')^*\mathcal{O}_T.$$

Using flat base change,

$$\begin{aligned}\mathbf{R}t'_*\mathcal{O}_{Y\times_X Y\times_X T} &\cong \mathbf{R}t'_*\mathbf{L}(f'\circ p')^*\mathcal{O}_T\\ &\cong \mathbf{L}(f\circ p)^*\mathbf{R}t_*\mathcal{O}_T.\end{aligned}$$

Then we obtain:

$$\begin{aligned}&\mathbf{R}t'_*\mathcal{O}_{Y\times_X Y\times_X T} \otimes^{\mathbf{L}} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y\\ &\quad\cong \mathbf{L}(f\circ p)^*\mathbf{R}t_*\mathcal{O}_T \otimes^{\mathbf{L}} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y && \text{(use above)}\\ &\quad\cong \mathbf{R}(\Delta_f)_*\mathbf{L}\Delta_f^*\mathbf{L}(f\circ p)^*\mathbf{R}t_*\mathcal{O}_T && \text{(projection formula)}\\ &\quad\cong \mathbf{R}(\Delta_f)_*\mathbf{L}(f\circ p\circ\Delta_f)^*\mathbf{R}t_*\mathcal{O}_T && \text{(pseudofunctoriality)}\\ &\quad\cong \mathbf{R}(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T && (p\circ\Delta_f = 1_Y).\end{aligned}$$

Since t is affine, the natural morphism $t_*\mathcal{O}_T \to \mathbf{R}t_*\mathcal{O}_T$ is an isomorphism. Moreover, f is flat, and so the natural morphism $f^*\mathbf{R}t_*\mathcal{O}_T \to \mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is an isomorphism. Hence, $\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is concentrated in degree zero. As Δ_f is affine, the natural morphism $(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T \to \mathbf{R}(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is an isomorphism. Thus, $\mathbf{R}^j(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T \cong 0$ for all $j \neq 0$.

Recall the natural isomorphism $\mathbf{L}(t')^*\mathcal{O}_{Y\times_X Y} \cong \mathcal{O}_{Y\times_X Y\times_X T}$. There is a morphism of distinguished triangles

$$\begin{tikzcd}
\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \arrow[r] \arrow[d] & \mathbf{L}(t')^*\mathcal{O}_{Y\times_X Y} \arrow[r] \arrow[d] & \mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \arrow[r] \arrow[d] & \mathbf{L}(t')^*\mathcal{I}_{\Delta_f}[1] \arrow[d]\\
\mathcal{I}_{\Delta'_f} \arrow[r] & \mathcal{O}_{Y\times_X Y\times_X T} \arrow[r] & \mathbf{R}(\Delta'_f)_*\mathcal{O}_{Y\times_X T} \arrow[r] & \mathcal{I}_{\Delta'_f}[1].
\end{tikzcd}$$

Since we have shown that the two inner morphisms are isomorphisms, it follows that the morphism $\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \to \mathcal{I}_{\Delta'_f}$ is an isomorphism. □

Theorem 5.5 (cf. [HCS09, Proposition 2.16]). *Let X be a Noetherian algebraic space. Consider an elliptic X-fibration $f\colon Y \to X$. Then $\Phi_{\mathcal{I}_{\Delta_f}}$ induces an autoequivalence of $D_{\mathrm{qc}}(Y)$ where $\mathcal{I}_{\Delta_f}$ is the ideal sheaf of the diagonal $\Delta_f\colon Y \to Y \times_X Y$. In fact, $\Phi_{\mathcal{I}_{\Delta_f}}$ yields an autoequivalence of $D^b_{\mathrm{coh}}(Y)$.*

Proof. We start with an observation. In this paragraph, fix $p \in |X|$. Choose a representative $\operatorname{Spec}(k) \to X$ of p such that $Y \times_X \operatorname{Spec}(k)$ is geometrically connected, of Krull dimension one, genus one, and with trivial dualizing sheaf. Let $\overline{k}$ be an algebraic closure of k. Since $Y \times_X \operatorname{Spec}(k)$ is geometrically integral, [Sta26, Tag 0FD2] shows that

$$H^0(Y \times_X \operatorname{Spec}(\overline{k}), \mathcal{O}_{Y\times_X \operatorname{Spec}(\overline{k})}) = \overline{k}.$$

By base change, $Y \times_X \operatorname{Spec}(\overline{k})$ is proper and Gorenstein over $\overline{k}$, and so [Sta26, Tag 0A26] implies $Y \times_X \operatorname{Spec}(\overline{k})$ is also projective over $\overline{k}$. Moreover, from [Sta26, Tag 0BY9], $Y \times_X \operatorname{Spec}(\overline{k})$ has genus one. Using [Sta26, Tag 0FW1], we see that $Y \times_X \operatorname{Spec}(\overline{k})$ has trivial dualizing sheaf. Thus, for every $p \in |X|$, we have a representative of p from an algebraically closed field such that the fiber of f is projective, geometrically connected, of Krull dimension one, genus one, and with trivial dualizing sheaf. Denote by K' the kernel of the integral transform from Proposition 4.17 which is right adjoint to $\Phi_{\mathcal{I}_{\Delta_f}}$ on D_{qc}.

Now, back to the proof. Since f is proper, the diagonal Δ_f is a closed immersion. Hence, $\mathcal{I}_{\Delta_f}$ is the negative shift of the cone of the unit $\mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y$. Note that Lemma 5.1 shows that $\mathcal{I}_{\Delta_f}$ is relatively perfect over Y. By Proposition 4.17, K' is bounded and pseudocoherent. Fix a closed point $p \in |X|$. Choose a representative $t\colon \operatorname{Spec}(k) \to X$ of p with k algebraically closed representatives constructed in the first paragraph. From Proposition 4.17, we know that $\Phi_{\mathbf{L}(t')^*\mathcal{I}_{\Delta_f}}$ and $\Phi_{\mathbf{L}(t')^*K'}$ remain an adjoint pair.

By Proposition 5.4, we have $\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \cong \mathcal{I}_{\Delta'_f}$. Consequently, we have that $\Phi_{\mathbf{L}(t')^*\mathcal{I}_{\Delta_f}}$ is naturally isomorphic to $\Phi_{\mathcal{I}_{\Delta'_f}}$ as endofunctors on $D^b_{\mathrm{coh}}(Y \times_X \operatorname{Spec}(k))$. Since geometric integrality of the fibers implies irreducible components have the same Krull dimension (i.e. equidimensional fibers), [HCS09, Proposition 2.16 & §3.4.1] implies that $\Phi_{\mathcal{I}_{\Delta'_f}}$ is an autoequivalence of $D^b_{\mathrm{coh}}(Y \times_X \operatorname{Spec}(k))$. In fact, as the fibers are schemes, [GLMP25, Lemma 2.10] ensures that $\Phi_{\mathcal{I}_{\Delta'_f}}$ is an autoequivalence on D_{qc}. Since $\Phi_{\mathbf{L}(t')^*K'}$ is right adjoint, it must be an equivalence on D_{qc}. Again, [GLMP25, Lemma 2.10] implies $\Phi_{\mathbf{L}(t')^*K'}$ restricts to an equivalence on Perf and D^b_{coh}. This says that $\mathbf{L}(t')^*K'$ is relatively perfect over both factors of the projections for $Y \times_X Y \times_X \operatorname{Spec}(k) \to Y \times_X \operatorname{Spec}(k)$. By Proposition 4.8(3) and Theorem 4.9, K' is relatively perfect over both factors of the projections for $Y \times_X Y \to Y$. Now, using Theorem 4.34, the desired claim follows since we have checked K' is relatively perfect over both factors of the projections for $Y \times_X Y \to Y$ and the pairs $\Phi_{\mathbf{L}(t')^*\mathcal{I}_{\Delta_f}}$ and $\Phi_{\mathbf{L}(t')^*K'}$ form adjoint equivalences for all closed points. □

5.2. **Openness of loci**.

Definition 5.6. Consider Setup 4.1 with each f_i surjective. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over each Y_i. We say that Φ_K is **fully faithful** (resp.

an **adjoint equivalence**) **at** $p \in |S|$ if $\mathbf{L}(t')^*K$ is such on D_{qc} for some representative t of p. Denote by $\mathrm{fm}(K; Y_1, Y_2, S)$ (resp. $\mathrm{FM}(K; Y_1, Y_2, S)$) the collection of $p \in |S|$ where Φ_K is fully faithful (resp. an adjoint equivalence) at p. If $Y_1 = Y_2 = Y$, we write this as $\mathrm{fm}(K; Y, S)$ (resp. $\mathrm{FM}(K; Y, S)$). In the case $S = \mathrm{Spec}(R)$ is an affine scheme, then we simply write $\mathrm{fm}(K; Y_1, Y_2, R)$ and $\mathrm{fm}(K; Y, R)$ (resp., $\mathrm{FM}(K; Y_1, Y_2, R)$ and $\mathrm{FM}(K; Y, R)$).

Lemma 5.7. *Consider Setup 4.1 with each f_i surjective. Let $K \in D_{qc}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over each Y_i. The condition that Φ_K is fully faithful (resp. an adjoint equivalence) at some $p \in |S|$ is independent of the representative of p.*

Proof. Choose some $p \in \mathrm{fm}(K; Y_1, Y_2, S)$, and let $t\colon \mathrm{Spec}(k) \to S$ be a representative where fully faithfulness holds. Pick any representative $h\colon \mathrm{Spec}(k') \to S$ of p. There exist a field ℓ and a commutative diagram,

$$\begin{tikzcd} \mathrm{Spec}(\ell) \arrow[r, "a"] \arrow[d, "b"'] & \mathrm{Spec}(k') \arrow[d, "h"] \\ \mathrm{Spec}(k) \arrow[r, "t"'] & S. \end{tikzcd}$$

Note that a, b are affine and faithfully flat. The hypothesis is that fully faithfulness occurs after base change along t. We can apply Proposition 4.33 to realize fully faithfulness occurs after base change along $t \circ b$. Since a is affine and faithfully flat, Proposition 4.30 implies fully faithfulness occurs after base change along h (e.g. use that $h \circ a = t \circ b$).

Lastly, let $p \in \mathrm{FM}(K; Y_1, Y_2, S)$. Choose a representative $t\colon \mathrm{Spec}(k) \to S$ for which the adjoint equivalence occurs. By Corollary 4.18, there exists $K' \in D^b_{coh}(Y_1 \times_S Y_2)$ such that Φ_K is a left adjoint of $\Phi_{K'}$ on D_{qc}. As $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$ form an adjoint equivalence, $\Phi_{\mathbf{L}(t')^*K'}$ is an exact equivalence. Hence, $\Phi_{\mathbf{L}(t')^*K'}$ restricts to an exact equivalence on the compacts (i.e. Perf). This means $\mathbf{L}(t')^*K'$ is relatively perfect over $Y_1 \times_S \mathrm{Spec}(k)$. Pick any representative $h\colon \mathrm{Spec}(k') \to S$ of p. There exist a field ℓ and a commutative diagram,

$$\begin{tikzcd} \mathrm{Spec}(\ell) \arrow[r, "a"] \arrow[d, "b"'] & \mathrm{Spec}(k') \arrow[d, "h"] \\ \mathrm{Spec}(k) \arrow[r, "t"'] & S. \end{tikzcd}$$

Note that a, b are affine and faithfully flat. By Lemma 4.7, the derived pullback of $\mathbf{L}(t')^*K'$ along natural base change obtained from b is relatively perfect over $Y_1 \times_S \mathrm{Spec}(\ell)$. Applying Proposition 4.8(1), we the derived pullback of K' along the natural morphism obtained by base change along h is relatively perfect over $Y_1 \times_S \mathrm{Spec}(k')$. This shows the derived pullback of K' is relatively $Y_1 \times_S \mathrm{Spec}(k')$-perfect for all representatives $\mathrm{Spec}(k') \to S$ of p. Applying Lemma 4.29, we can argue like fully faithful case above to finish the proof. □

Theorem 5.8. *Consider Setup 4.1 with each f_i surjective. Let $K \in D_{qc}(Y_1 \times_S Y_2)$ be pseudocoherent and relatively perfect over each Y_i. Then the collection of $p \in |S|$, for which $\Phi_{\mathbf{L}(t')^*K}$ along some representative t of p is an adjoint equivalence (resp. fully faithful) on D_{qc}, is an open subset of $|S|$.*

Proof. By Corollary 4.18, there exists $K' \in D^b_{coh}(Y_1 \times_S Y_2)$ such that Φ_K is a left adjoint of $\Phi_{K'}$ on D_{qc}. In fact, the adjoint pair restricts to D^b_{coh}. Denote by $\epsilon\colon \Phi_K \circ \Phi_{K'} \to 1$ the counit and $\eta\colon 1 \to \Phi_{K'} \circ \Phi_K$ the unit for this adjunction. Choose compact generators G_i

for $D_{\mathrm{qc}}(Y_i)$. Set $C_1 := \mathrm{cone}(\eta_{G_1})$ and $C_2 := \mathrm{cone}(\epsilon_{G_2})$. We claim that

$$f_1(\mathrm{supp}(C_1)) = |S| \setminus \mathrm{fm}(K; Y_1, Y_2, S)$$

and

$$|S| \setminus \mathrm{FM}(K; Y_1, Y_2, S) = f_1(\mathrm{supp}(C_1)) \cup f_2(\mathrm{supp}(C_2)).$$

Since Φ_K and $\Phi_{K'}$ restrict to an adjoint pair on D^b_{coh}, each C_1 and C_2 are bounded pseudocoherent. Thus, as each f_i is a closed morphism, the desired claim would follow.

We prove the desired claim. For any $p \in |S|$ and representative $t\colon \mathrm{Spec}(k) \to S$ of p, (4.12) and (4.15) imply

$$\mathbf{L}t_1^* C_1 \cong \mathrm{cone}(\eta'_{\mathbf{L}t_1^* G_1})$$

and

$$\mathbf{L}t_2^* C_2 \cong \mathrm{cone}(\epsilon'_{\mathbf{L}t_2^* G_2}).$$

where $\epsilon'\colon \Phi_{\mathbf{L}(t')^*K} \circ \Phi_{\mathbf{L}(t')^*K'} \to 1$ is the counit and $\eta'\colon 1 \to \Phi_{\mathbf{L}(t')^*K'} \circ \Phi_{\mathbf{L}(t')^*K}$ the unit for the adjoint pair $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$. See Proposition 4.17. Since $\mathbf{L}t_i^* G_i$ are compact generators of $D_{\mathrm{qc}}(Y_i \times_S \mathrm{Spec}(k))$, Lemma 4.28 gives

$$\mathbf{L}t_1^* C_1 \cong 0 \iff \Phi_{\mathbf{L}(t')^*K} \text{ is fully faithful}$$

and

$$\mathbf{L}t_1^* C_1 \cong 0 \text{ and } \mathbf{L}t_2^* C_2 \cong 0 \iff \Phi_{\mathbf{L}(t')^*K} \text{ is an adjoint equivalence.}$$

Consequently, the claim follows from Lemmas 2.5 and 5.7. □

Example 5.9. Let S be a Dedekind scheme with generic point η. Suppose C is a geometrically connected elliptic curve over $\kappa(\eta)$. There exists a proper flat morphism $f\colon Y \to S$ from a scheme of Krull dimension two satisfying $Y \times_S \mathrm{Spec}(\kappa(\eta)) \cong C$. See e.g. [Liu02, §10.1.1]. Denote by $\mathcal{I}_{\Delta_f}$ the ideal sheaf of the diagonal $\Delta_f\colon Y \to Y \times_S Y$. Since the generic fiber of f is a geometrically connected elliptic curve, Theorem 5.5 implies $\Phi_{\mathcal{I}_{\Delta_f}}$ induces an adjoint autoequivalence at η. Consequently, by Theorem 5.8, $\Phi_{\mathcal{I}_{\Delta_f}}$ induces adjoint autoequivalences at all points of S but possibly finitely many.

Appendix A. Preferred equivalence classes

We prove versions of [DLMP25, Proposition 6.6], [Lan26, Proposition 3.1], and [HR23, Proposition 4.2] for algebraic spaces. The difference is that loc. cit. takes place on the lisse-étale site, and so we prove it on the small étale site for the sake of completeness. This requires a placeholder:

(A.1) An algebraic space X satisfies **P.E.C.** if for any quasi-compact open immersion $U \to X$ with complement Z, the standard t-structure on $D_{\mathrm{qc},Z}(X)$ is in the preferred equivalence class.

Lemma A.1. *Consider a recollement of triangulated categories*

$$\begin{tikzcd} \mathcal{D}_Z \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} .$$

Let G_U be a compact generator $\mathcal{D}_U$. Assume that $\mathcal{D}$ admits a compact generator G' such that $\mathrm{Hom}(G'[-n], G') = 0$ *for $n \gg 0$ and $j^*G' \in \mathcal{D}_U^{\leq m}$ for some $m > 0$ (here, $\mathcal{D}_U^{\leq 0}$ is the aisle of the*

t-structure on $\mathcal{D}_U$ compactly generated by G_U). There exists a compact generator $G \in \mathcal{D}$, which can be taken to be $G'[m] \oplus j_! G_U$, such that the t-structure on $\mathcal{D}$ compactly generated by G is the gluing of the t-structure on $\mathcal{D}_Z$ compactly generated by i^*G and the t-structure on $\mathcal{D}_U$ compactly generated by G_U. In particular, the glued t-structure is in the preferred equivalence class.

Proof. This was recorded in [DLMP25, Lemma 5.6]. The proof is argued verbatim to [BNP23, Lemma 3.11] where in loc. cit. one can replace the condition $\mathcal{D}_U$ 'weakly approximable' by the hypotheses that $j^*G' \in \mathcal{D}_U^{\leq m}$. □

Lemma A.2. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Consider a quasi-compact open immersion $j\colon U \to X$. Set $Z := |X| \setminus |U|$. Then*

$$\mathbf{L}f^*(D_{\mathrm{qc},Z}(X)) \subseteq D_{\mathrm{qc},f^{-1}(Z)}(Y).$$

Proof. Choose $E \in D_{\mathrm{qc},Z}(X)$. Set $f'\colon f^{-1}(U) \to U$ to be the base change of j along f. As $\mathbf{L}j^*E \cong 0$, one has $\mathbf{L}(j \circ f')^*E \cong 0$. Define $j'\colon f^{-1}(U) \to Y$ to be the open immersion obtained by base change. Hence, $\mathbf{L}(f \circ j')^*E \cong 0$, implying that $\mathbf{L}f^*E \in D_{\mathrm{qc},f^{-1}(Z)}(Y)$. □

Lemma A.3. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated algebraic spaces. Consider a quasi-compact open immersion $j\colon U \to X$. Set $Z := |X| \setminus |U|$. Then*

$$\mathbf{R}f_*(D_{\mathrm{qc},f^{-1}(Z)}(Y)) \subseteq D_{\mathrm{qc},Z}(X).$$

Proof. By [Sta26, Tag 0AEC], $D_{\mathrm{qc},f^{-1}(Z)}(Y)$ is compactly generated by a single object G. Recall that $\mathbf{R}f_*$ preserves small coproducts. Applying [Sta26, Tag 09SN], we can reduce to checking that $\mathbf{R}f_*G \in D_{\mathrm{qc},Z}(X)$. As $f^{-1}(Z) = |Y| \setminus |f^{-1}(U)|$, it follows that $\mathbf{L}(j')^*G \cong 0$ where $j'\colon Y \times_X U \to Y$ is the projection from the fiber product obtained base changing f along j. Denote $f'\colon Y \times_X U \to U$ for the other projection. By flat base change, it follows that

$$\mathbf{L}j^*\mathbf{R}f_*G \cong \mathbf{R}f'_*\mathbf{L}(j')^*G \cong 0.$$

This completes the proof. □

Lemma A.4. *Let X be a quasi-compact quasi-separated algebraic space. Let $j\colon U \to X$ be a quasi-compact open immersion. Let $Z := |X| \setminus |U|$. There exists a recollement*

$$\begin{tikzcd} D_{\mathrm{qc}}(U) \arrow[r, "\mathbf{R}j_*"] & D_{\mathrm{qc}}(X) \arrow[l, bend right, "\mathbf{L}j^*"'] \arrow[l, bend left, "j^\times"] \arrow[r, "i^!"] & D_{\mathrm{qc},Z}(X) \arrow[l, bend right, "i_*"'] \arrow[l, bend left, "i^*"] \end{tikzcd}$$

where i_* is the natural inclusion and $j^\times$ the right adjoint of $\mathbf{R}j_*$. Additionally, given a closed $W \subseteq |X|$ subset with quasi-compact complement, there exists a recollement

$$\begin{tikzcd} D_{\mathrm{qc},|U|\cap W}(U) \arrow[r, "\mathbf{R}j_*"] & D_{\mathrm{qc},W}(X) \arrow[l, bend right, "\mathbf{L}j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "i^!"] & D_{\mathrm{qc},W\cap Z}(X) \arrow[l, bend right, "i_*"'] \arrow[l, bend left, "i^*"] \end{tikzcd}$$

where i_* is the natural inclusion.

Proof. By [Sta26, Tag 0AEC], $D_{\mathrm{qc},|U|\cap W}(U)$ is compactly generated. We show the second claim because the first claim is similar (note that one obtains $j^!$ is naturally isomorphic to $j^\times$ by uniqueness of adjoints). By Lemma A.2, the restriction of $\mathbf{L}j^* D_{\mathrm{qc},W}(X) \subseteq D_{\mathrm{qc},|U|\cap W}(U)$. Moreover, Lemma A.3 says $\mathbf{R}j_* D_{\mathrm{qc},|U|\cap W}(U) \subseteq D_{\mathrm{qc},W}(X)$. Hence, the adjoint pair $\mathbf{L}j^*$ and $\mathbf{R}j_*$ restricts to $D_{\mathrm{qc},W}(X)$ and $D_{\mathrm{qc},|U|\cap W}(U)$. There exists a Verdier localization sequence,

$$D_{\mathrm{qc},W\cap Z}(X) \xrightarrow{i_*} D_{\mathrm{qc},W}(X) \xrightarrow{\mathbf{L}j^*} D_{\mathrm{qc},|U|\cap W}(U).$$

To see, use that the counit of $\mathbf{L}j^*$ and $\mathbf{R}j_*$ is a natural isomorphism on D_{qc}, which implies that $\mathbf{L}j^*\colon D_{\mathrm{qc},W}(X) \to D_{\mathrm{qc},|U|\cap W}(U)$ is essentially surjective. To see that fully faithfulness holds, use [GZ67, §I. Proposition 1.3] and [Kra10, Lemma 4.3.1]. As $\mathbf{R}j_*$ is right adjoint to $\mathbf{L}j^*$, [GP18, Lemma 2.2(ii)] implies i_* admits a right adjoint. We denote it by $i^!$ (see also [CPS88b, Theorem 1.1] and [CPS88a, Theorem 2.1]). Hence, we obtain another Verdier localization sequence,

$$D_{\mathrm{qc},Z\cap W}(X) \xleftarrow{i^!} D_{\mathrm{qc},W}(X) \xleftarrow{\mathbf{R}j_*} D_{\mathrm{qc},|U|\cap W}(U).$$

Since $D_{\mathrm{qc},|U|\cap W}(U)$ is compactly generated and $\mathbf{R}j_*$ preserves small coproducts, [BDS16, Corollary 2.3] implies the restriction of $\mathbf{R}j_*$ admits a right adjoint. Applying [GP18, Lemma 2.2(ii)], $i^!$ admits a right adjoint as well, which we denote by i^*. Thus, we have the required data for a recollement; i.e. a Verdier localization sequence which is a localization and colocalization sequence in the sense of [Kra10, §4]. □

Remark A.5. In Lemma A.4, $i^!$ preserves small coproducts, and so, i_* preserves compact objects (see e.g. the proof of $\implies$ in [Nee96, Theorem 5.1]; which this fact does not require compact generation).

Lemma A.6. *Consider an elementary distinguished square $j\colon U \to X$ and $f\colon V \to X$ where j is an open immersion of quasi-compact quasi-separated algebraic spaces. Set $Z := |X| \setminus |U|$. Then the restrictions of $\mathbf{L}f^*$ and $\mathbf{R}f_*$ yield t-exact equivalences:*

(1) $D_{\mathrm{qc},f^{-1}(Z)}(V)$ and $D_{\mathrm{qc},Z}(X)$
(2) $D^b_{\mathrm{qc},f^{-1}(Z)}(V)$ and $D^b_{\mathrm{qc},Z}(X)$.

Proof. Denote by ϵ and η respectively the counit and unit of the adjoint pair $\mathbf{L}f^*$ and $\mathbf{R}f_*$. Applying [Sta26, Tag 08GG], ϵ and η are isomorphisms when restricted to objects with the prescribed support. It follows that $\mathbf{L}f^*$ and $\mathbf{R}f_*$ restrict to an equivalence $D_{\mathrm{qc},f^{-1}(Z)}(V)$ and $D_{\mathrm{qc},Z}(X)$. Since f is étale, $\mathbf{L}f^*$ is t-exact, whereas the restricted adjoint pair being an equivalence implies the restriction of $\mathbf{R}f^*$ is t-exact. By t-exactness, the restricted functors preserve bounded cohomology, and so the second claim follows. □

Lemma A.7. *Consider an elementary distinguished square $j\colon U \to X$ and $f\colon Y \to X$ of quasi-compact quasi-separated algebraic spaces. If U and Y satisfy P.E.C., then so does X.*

Proof. Define $Z := |X| \setminus |U|$. Choose $V \to X$ a quasi-compact open immersion. Set $W := |X| \setminus |V|$. By Lemma A.4, there exists a recollement

$$\begin{tikzcd}[column sep=large] D_{\mathrm{qc},|U|\cap W}(U) \arrow[r, "\mathbf{R}j_*"] & D_{\mathrm{qc},W}(X) \arrow[l, bend right, "\mathbf{L}j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "i^!"] & D_{\mathrm{qc},Z\cap W}(X) \arrow[l, bend right, "i_*"'] \arrow[l, bend left, "i^*"] \end{tikzcd}$$

where i_* is the natural inclusion. We begin by showing there exists an $a \geq 0$ such that $i^! D^{\leq 0}_{\mathrm{qc},W}(X) \subseteq D^{\leq a+1}_{\mathrm{qc},Z\cap W}(X)$. Fix $E \in D^{\leq 0}_{\mathrm{qc},W}(X)$. As j is flat and quasi-affine, $\mathbf{L}j^*E \in D^{\leq 0}_{\mathrm{qc},|U|\cap W}(U)$ and $\mathbf{R}j_* D^{\leq 0}_{\mathrm{qc},|U|\cap W}(U) \subseteq D^{\leq a}_{\mathrm{qc},W}(X)$ for some $a \geq 0$. To see, note that flatness implies $\mathbf{L}j^*$ is t-exact with respect to the standard t-structures, whereas open immersions ensure the existence of such an a (e.g. one can reduce to schemes after base changing along an étale presentation, use that open immersions are representable by schemes, and apply [LH09, Proposition 3.9.2]). The recollement above gives a distinguished triangle

$$i_*i^!E \to E \to \mathbf{R}j_*\mathbf{L}j^*E \to i_*i^!E[1]. \tag{A.2}$$

Note that $\mathbf{R}j_*\mathbf{L}j^*E \in D^{\leq a}_{\mathrm{qc},W}(X)$ and

$$E \in D^{\leq 0}_{\mathrm{qc},W}(X) \subseteq D^{\leq 0}_{\mathrm{qc},W}(X)[-a] = D^{\leq a}_{\mathrm{qc},W}(X).$$

From the distinguished triangle above, it follows $i_*i^!E[1] \in D^{\leq a}_{\mathrm{qc},W}(X)$. Hence, we obtain that $i^!E \in D^{\leq a+1}_{\mathrm{qc},Z\cap W}(X)$ (i_* is the inclusion).

Let us return to the main proof. As $D_{\mathrm{qc},W}(X)$ is singly compactly generated, there exists $P \in D_{\mathrm{qc},W}(X) \cap \mathrm{Perf}(X)$ such that $\mathbf{L}j^*P$ compactly generates $D_{\mathrm{qc},|U|\cap W}(U)$. To see, use that $\mathbf{R}j_*$ is conservative, and so $\mathbf{L}j^*P$ is a compact generator if P is such for $D_{\mathrm{qc},W}(X)$ (see e.g. Lemma B.3). By Lemma A.6, $\mathbf{L}f^*$ and $\mathbf{R}f_*$ restrict to yield equivalences between $D_{\mathrm{qc},f^{-1}(Z)}(V)$ and $D_{\mathrm{qc},Z}(X)$. These equivalences are t-exact with respect to the standard t-structures. Choose P_Z a compact generator for $D_{\mathrm{qc},Z\cap W}(X)$. As $i^!$ preserves coproducts (e.g. it admits a right adjoint), it follows that i_* preserves compacts, which implies $i_*P_Z \in D_{\mathrm{qc},W}(X) \cap \mathrm{Perf}(X)$.

Note that $\mathrm{Hom}(P[-n],P) = 0$ for $n \gg 0$. After shifting, we can assume that $P \oplus i_*P_Z \in D^{\leq 0}_{\mathrm{qc}}(X)$. Applying the observation for some $a \geq 0$, $i^!P \in D^{\leq a+1}_{\mathrm{qc},Z\cap W}(X)$. By Lemma A.1, the glued t-structure on $D_{\mathrm{qc},W}(X)$, of the aisles on respectively $D_{\mathrm{qc},|U|\cap W}(U)$ and $D_{\mathrm{qc},Z\cap W}(X)$ compactly generated by $\mathbf{L}j^*P$ and P_Z, is compactly generated by $P[a+1] \oplus i_*P_Z$.

We show there exists an $N \geq 0$ such that

$$D^{\leq -N}_{\mathrm{qc},W}(X) \subseteq \mathrm{Coprod}(\{(P \oplus i_*P_Z)[s] \mid s \geq 0\}) \subseteq D^{\leq N}_{\mathrm{qc},W}(X)$$

which shows our desired claim as $(P \oplus i_*P_Z)$ is also a compact generator. As $P \oplus i_*P_Z \in D^{\leq 0}_{\mathrm{qc}}(X)$, we obtain

$$\mathrm{Coprod}(\{(P \oplus i_*P_Z)[s] \mid s \geq 0\}) \subseteq D^{\leq t}_{\mathrm{qc},W}(X) = D^{\leq 0}_{\mathrm{qc},W}(X)[-t]$$

for all $t \geq 0$. We are left to check that there exists an $N \geq 0$ such that $D^{\leq -N}_{\mathrm{qc},W}(X) \subseteq \mathrm{Coprod}(\{(P \oplus i_*P_Z)[s] \mid s \geq 0\})$. It suffices to find some $N \geq 0$ such that

$$\begin{aligned} \mathbf{L}j^* D^{\leq -N}_{\mathrm{qc},W}(X) &\subseteq \mathrm{Coprod}(\{\mathbf{L}j^*P[s] \mid s \geq 0\}) \\ i^! D^{\leq -N}_{\mathrm{qc},W}(X) &\subseteq \mathrm{Coprod}(\{P_Z[s] \mid s \geq 0\}). \end{aligned}$$

Indeed, the glued t-structure yields,

$$\begin{aligned} D^{\leq -N}_{\mathrm{qc},W}(X) &\subseteq \mathrm{Coprod}(\{(P[a+1] \oplus i_*P_Z)[s] \mid s \geq 0\}) \\ &\subseteq \mathrm{Coprod}(\{(P \oplus i_*P_Z)[s] \mid s \geq 0\}). \end{aligned}$$

Now, applying our hypothesis on U and Y, for all $m \gg 0$, it follows that

$$D^{\leq -m}_{\mathrm{qc},|U|\cap W}(U) \subseteq \operatorname{Coprod}(\{\mathbf{L}j^*P[s] \mid s \geq 0\}) \quad \text{and}$$
$$D^{\leq -m}_{\mathrm{qc},Z\cap W}(X) \subseteq \operatorname{Coprod}(\{P_Z[s] \mid s \geq 0\}).$$

We explain the last inclusion. By Lemma A.6, we know that $\mathbf{L}f^*P_Z$ is a compact generator for $D_{\mathrm{qc},f^{-1}(W\cap Z)}(Y)$. As Y satisfies P.E.C., $\operatorname{Coprod}(\{\mathbf{L}f^*P_Z[s] \mid s \geq 0\})$ and $D^{\leq 0}_{\mathrm{qc},f^{-1}(W\cap Z)}(Y)$ are in the same equivalence class. Applying the t-exact equivalences of Lemma A.6, the inclusion follows. Lastly, taking $N := m + a + 1 \geq 0$,

$$\begin{aligned}\mathbf{L}j^*D^{\leq -N}_{\mathrm{qc},W}(X) &\subseteq D^{\leq -N}_{\mathrm{qc},|U|\cap W}(U)\\ &\subseteq D^{\leq -m}_{\mathrm{qc},|U|\cap W}(U)\\ &\subseteq \operatorname{Coprod}(\{\mathbf{L}j^*P[s] \mid s \geq 0\}).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}i^!D^{\leq -N}_{\mathrm{qc},W}(X) &\subseteq D^{\leq -m}_{\mathrm{qc},Z\cap W}(X)\\ &\subseteq \operatorname{Coprod}(\{P_Z[s] \mid s \geq 0\}).\end{aligned}$$

This completes the proof. □

Lemma A.8. *Any quasi-compact quasi-separated algebraic space satisfies P.E.C.*

Proof. The claim is known for schemes [Nee24, Theorem 3.2(iii)]. Hence, the claim holds for the schematic locus of a quasi-compact quasi-separated algebraic space. We apply the induction principle [Sta26, Tag 09IT]. By Lemma A.7, we finish the proof. □

Appendix B. Fiber products

We prove a version of [Nee23, Corollary 5.10] for the small étale site.

Lemma B.1. *Let $f_i\colon Y_i \to S$ be morphisms of quasi-compact quasi-separated algebraic spaces. If P_i compactly generates $D_{\mathrm{qc}}(Y_i)$ for each i, then $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ is compactly generated by $\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2$ where $g_i\colon Y_1 \times_S Y_2 \to Y_i$ are the natural projections.*

Lemma B.2. *Lemma B.1 is true if S is a quasi-compact quasi-separated scheme.*

Proof. Consider an $E \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ which satisfies

$$\operatorname{Hom}(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2, E[n]) = 0$$

for all $n \in \mathbb{Z}$. We claim that $E \cong 0$. In such a case, $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ is compactly generated by $\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2$, which would complete the proof. Towards that end, choose étale presentations $s_i\colon U_i \to Y_i$ from affine schemes. Consider the commutative diagram

$$\begin{tikzcd}
U_1 \times_S U_2 \arrow[r, "s_1''"] \arrow[d, "s_2''"'] \arrow[dr, "s"] & Y_1 \times_S U_2 \arrow[r, "g_2'"] \arrow[d, "s_2'"] & U_2 \arrow[d, "s_2"] \\
U_1 \times_S Y_2 \arrow[r, "s_1'"'] \arrow[d, "g_1'"'] & Y_1 \times_S Y_2 \arrow[r, "g_2"] \arrow[d, "g_1"] & Y_2 \arrow[d, "f_2"] \\
U_1 \arrow[r, "s_1"'] & Y_1 \arrow[r, "f_1"'] & S
\end{tikzcd}$$

whose squared faces are fibered.

Using adjunctions, there is a string of isomorphisms for any $n \in \mathbb{Z}$,

$$\begin{aligned}
&\operatorname{Hom}(\mathbf{L}g_1^* P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^* P_2, E[n]) \\
&\cong \operatorname{Hom}(\mathbf{L}g_1^* P_1, \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E[n])) && \text{(tensor/hom)} \\
&\cong \operatorname{Hom}(P_1, \mathbf{R}(g_1)_* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E)[n]) && \text{(pull/push)}
\end{aligned}$$

From $D_{\mathrm{qc}}(Y_1)$ being compactly generated by P_1, it follows that $\mathbf{R}(g_1)_* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E) \cong 0$. Then, by flat base change,

$$0 \cong \mathbf{L}s_1^* \mathbf{R}(g_1)_* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E) \cong \mathbf{R}(g_1')_* \mathbf{L}(s_1')^* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E).$$

However, once again by adjunctions, for all integers n,

$$\begin{aligned}
&0 \\
&\cong \operatorname{Hom}(\mathcal{O}_{U_1}, \mathbf{R}(g_1')_* \mathbf{L}(s_1')^* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E)[n]) \\
&\cong \operatorname{Hom}(\mathbf{L}(g_1')^* \mathcal{O}_{U_1}, \mathbf{L}(s_1')^* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E)[n]) && \text{(pull/push)} \\
&\cong \operatorname{Hom}(\mathcal{O}_{U_1 \times_S Y_2}, \mathbf{L}(s_1')^* \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^* P_2, E)[n]) && (\mathbf{L}(g_1')^* \mathcal{O}_{U_1} = \mathcal{O}_{U_1 \times_S Y_2}) \\
&\cong \operatorname{Hom}(\mathcal{O}_{U_1 \times_S Y_2}, \mathbf{R}\mathcal{H}om(\mathbf{L}(s_1')^* \mathbf{L}g_2^* P_2, \mathbf{L}(s_1')^* E)[n]) && \text{(Lemma 2.11)} \\
&\cong \operatorname{Hom}(\mathcal{O}_{U_1 \times_S Y_2}, \mathbf{R}\mathcal{H}om(\mathbf{L}(g_2 \circ s_1')^* P_2, \mathbf{L}(s_1')^* E)[n]) && \text{(pseudofunctoriality)} \\
&\cong \operatorname{Hom}(\mathbf{L}(g_2 \circ s_1')^* P_2, \mathbf{L}(s_1')^* E[n]) && \text{(tensor/hom)} \\
&\cong \operatorname{Hom}(P_2, \mathbf{R}(g_2 \circ s_1')_* \mathbf{L}(s_1')^* E[n]) && \text{(pull/push).}
\end{aligned}$$

So, $D_{\mathrm{qc}}(Y_2)$ being compactly generated by P_2 tells us $\mathbf{R}(g_2 \circ s_1')_* \mathbf{L}(s_1')^* E \cong 0$. As $f_1 \circ s_1$ is quasi-affine (see e.g. [Sta26, Tag 01SP]), base change implies $g_2 \circ s_1'$ must be such. In such a case, $\mathbf{R}(g_2 \circ s_1')_* \mathbf{L}(s_1')^* E \cong 0$ implies $\mathbf{L}(s_1')^* E \cong 0$ (see Lemma 3.2), and hence, $\mathbf{L}s^* E \cong 0$. However, s is a faithfully flat morphism, so $E \cong 0$ as desired. □

Lemma B.3. *Let $f \colon Y \to X$ be a quasi-affine morphism of quasi-compact quasi-separated algebraic spaces. Then $\mathbf{L}f^* G$ compactly generates $D_{\mathrm{qc}}(Y)$ whenever $G \in D_{\mathrm{qc}}(X)$ does such for $D_{\mathrm{qc}}(X)$.*

Proof. By Lemma 3.2, $\mathbf{R}f_*$ is conservative. Now, apply [Lan26, Corollary 3.6]. □

Proof of Lemma B.1. Let $s \colon U \to S$ be an étale presentation from an affine scheme. Consider the commutative diagram

$$\begin{tikzcd}
Y_1 \times_S Y_2 \times_S U \arrow[rr, "g_2'"] \arrow[d, "s'"] \arrow[dr, "g_1'"] & & Y_2 \times_S U \arrow[d, dashed, "s_2"] \arrow[dr, "f_2'"] & \\
Y_1 \times_S Y_2 \arrow[rr, dashed, "g_2"] \arrow[dr, "g_1"] & Y_1 \times_S U \arrow[rr, "f_1'"] \arrow[d, "s_1"] & Y_2 \arrow[dr, dashed, "f_2"] & U \arrow[d, "s"] \\
& Y_1 \arrow[rr, "f_1"] & & S
\end{tikzcd}$$

whose faces are fibered. From Lemma 2.13, we see that s is quasi-affine. By base change, s' and each s_i must be smooth, quasi-affine, and surjective morphisms.

Now, for each i, Lemma B.3 tells us $\mathbf{L}s_i^*P_i$ compactly generates $D_{\mathrm{qc}}(Y_i \times_S U)$. By Lemma B.2, $\mathbf{L}(g_1')\mathbf{L}s_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}(g_2')\mathbf{L}s_2^*P_2$ also compactly generates $D_{\mathrm{qc}}(Y_1 \times_S Y_2 \times_S U)$. Moreover, from the isomorphisms,

$$\begin{aligned}\mathbf{L}(g_1')\mathbf{L}s_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}(g_2')\mathbf{L}s_2^*P_2 &\cong \mathbf{L}(g_1 \circ s')^*P_1 \otimes^{\mathbf{L}} \mathbf{L}(g_2 \circ s')^*P_2 \\ &\cong \mathbf{L}(s')^*(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2),\end{aligned}$$

we see that $\mathbf{L}(s')^*(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2)$ compactly generates $D_{\mathrm{qc}}(Y_1 \times_S Y_2 \times_S U)$.

Consider an $E \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ satisfying $\mathrm{Hom}(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2, E[n]) = 0$ for all $n \in \mathbb{Z}$. Using adjunctions, we see for all $n \in \mathbb{Z}$,

$$\begin{aligned}&\mathrm{Hom}(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2, E[n]) \\ &\cong \mathrm{Hom}(\mathbf{L}g_1^*P_1, \mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E[n])) && \text{(tensor/hom)} \\ &\cong \mathrm{Hom}(P_1, \mathbf{R}(g_1)_* \,\mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E)[n]) && \text{(pull/push)}\end{aligned}$$

As P_1 compactly generates $D_{\mathrm{qc}}(Y_1)$, it follows that $\mathbf{R}(g_1)_* \,\mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E) \cong 0$. Then, by flat base change,

$$0 \cong \mathbf{L}s_1^*\mathbf{R}(g_1)_* \,\mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E[n]) \cong \mathbf{R}(g_1')_*\mathbf{L}(s')^* \,\mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E).$$

Furthermore, from adjunction once again, for all $n \in \mathbb{Z}$,

$$\begin{aligned}0 &= \mathrm{Hom}(\mathbf{L}s_1^*P_1, \mathbf{R}(g_1')_*\mathbf{L}(s')^* \,\mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E[n])) \\ &= \mathrm{Hom}(\mathbf{L}(g_1')^*\mathbf{L}s_1^*P_1, \mathbf{L}(s')^* \,\mathbf{R}\mathcal{H}om(\mathbf{L}g_2^*P_2, E[n])) && \text{(pull/push)} \\ &= \mathrm{Hom}(\mathbf{L}(g_1')^*\mathbf{L}s_1^*P_1, \mathbf{R}\mathcal{H}om(\mathbf{L}(s')^*\mathbf{L}g_2^*P_2, \mathbf{L}(s')^*E[n])) && \text{(Lemma 2.11)} \\ &= \mathrm{Hom}(\mathbf{L}(g_1')^*\mathbf{L}s_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}(s')^*\mathbf{L}g_2^*P_2, \mathbf{L}(s')^*E[n]) && \text{(tensor/hom)} \\ &= \mathrm{Hom}(\mathbf{L}(s')^*(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2), \mathbf{L}(s')^*E[n]).\end{aligned}$$

Consequently, $\mathbf{L}(s')^*E \cong 0$ because $D_{\mathrm{qc}}(Y_1 \times_S Y_2 \times_S U)$ is compactly generated by $\mathbf{L}(s')^*(\mathbf{L}g_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}g_2^*P_2)$. This implies $E \cong 0$ because s' is a smooth surjective morphism, and hence, completes the proof. □

Appendix C. Duality for spaces

We record the analog of Grothendieck duality for Noetherian algebraic spaces appearing in our work; see [Sta26, Tag 0E4V]. The results below are those of [Nee23], reformulated for the small étale site. Since the arguments in [Nee23] are purely formal, the proofs apply verbatim to the derived category of complexes with quasi-coherent cohomology on the small étale site.

Categories. Fix a base scheme S, regarded as an object of the big fppf site [Sta26, Tag 021R]. Let $(\mathrm{Sch}/S)_{fppf}$ be the big fppf site of S [Sta26, Tag 021S]. We define the $(2,1)$-category Sp of algebraic spaces over S as follows.

Every algebraic space X over S determines a category $\mathcal{S}_X$ fibered in sets over $(\mathrm{Sch}/S)_{fppf}$ [Sta26, Tags 04TM & 02Y2], and hence an object in the 2-category of categories fibered over $(\mathrm{Sch}/S)_{fppf}$ [Sta26, Tag 04S8].

A morphism of algebraic spaces $f\colon Y \to X$ induces a 1-morphism $\mathcal{S}_Y \to \mathcal{S}_X$ in the 2-category of categories fibered over $(\mathrm{Sch}/S)_{fppf}$. Thus algebraic spaces over S form a $(2,1)$-category, denoted Sp/S, whose 2-morphisms are invertible [Sta26, Tag 003S]. In particular, Sp/S is a 2-subcategory of the $(2,1)$-category of algebraic stacks over S [Sta26, Tag 03YP].

Denote by $(\mathrm{Sp}/S)_{Noeth,sep,fp}$ the 2-subcategory of Sp/S consisting of Noetherian algebraic spaces, separated finitely presented morphisms, and all 2-morphisms. Every 1-morphism $f\colon Y \to X$ of $(\mathrm{Sp}/S)_{Noeth,sep,fp}$ admits a Nagata compactification $f = p \circ j$ where $p\colon Y' \to X$ is proper and $j\colon Y \to Y'$ is an open immersion [CLO12, Ra071, Ra074]. Let TriCat be the 2-category whose objects are triangulated categories, 1-morphisms are exact functors, and 2-morphisms are natural transformations.

Functors. There exist two contravariant pseudofunctors $\mathbf{L}(-)^*, (-)^\times\colon (\mathrm{Sp}/S)_{Noeth,sep,fp} \to$ TriCat, and a functor $(-)^!\colon (\mathrm{Sp}/S)_{Noeth,sep,fp} \to$ TriCat which is contravariant and oplax. We define them as follows:

- On objects: $\mathbf{L}(X)^* = (X)^\times = (X)^! = D_{\mathrm{qc}}(X)$ for all objects X in $(\mathrm{Sp}/S)_{Noeth,sep,fp}$.
- On 1-morphisms: For a 1-morphism $f\colon Y \to X$ in $(\mathrm{Sp}/S)_{Noeth,sep,fp}$, $\mathbf{L}f^*$ is the derived pullback functor, $f^\times$ is the right adjoint to pushforward $\mathbf{R}f_*$, and $f^!$ is defined as in [Nee23, Construction 9.4] (if $f = p \circ j$ is a Nagata compactification, then $f^! := j^* \circ p^\times$).
- On 2-morphisms: Since $(\mathrm{Sp}/S)_{Noeth,sep,fp}$ is a $(2,1)$-category, each 2-morphism is invertible, and so we obtain natural isomorphisms between functors.

The functors $\mathbf{L}(-)^*$ and $(-)^\times$ are pseudofunctorial; that is, if $f\colon Z \to Y$ and $g\colon Y \to X$ are composable 1-morphisms, then there exist natural isomorphisms $\tau(f,g)\colon \mathbf{L}(g\circ f)^* \implies \mathbf{L}f^* \circ \mathbf{L}g^*$ and $\delta(f,g)\colon (g \circ f)^\times \implies f^\times \circ g^\times$. In contrast, $(-)^!$ is not in general pseudofunctorial. Rather, $(-)^!$ is oplax, equipped with natural morphisms $\rho(f,g)\colon (g \circ f)^! \implies f^! \circ g^!$. Moreover, $(-)^\times$ and $(-)^!$ are oplax modules over $\mathbf{L}(-)^*$. This means there exist oplax natural transformations $\xi\colon \mathbf{L}(-)^* \times (-)^\times \to (-)^\times$ and $\sigma\colon \mathbf{L}(-)^* \times (-)^! \to (-)^!$ satisfying an associativity property. See [Nee23, Theorem 1.8(i to v)].

Relations. We now collect base change formulas and properties of these functors. The following results are [Nee23, Theorem 1.8.6 to 1.8.8 & Lemma 0.1].

Lemma C.1. *Let $f\colon Z \to Y$ and $g\colon Y \to X$ be a pair of composable 1-morphisms in $(\mathrm{Sp}/S)_{Noeth,sep,fp}$. Then the morphism $\rho(f,g)\colon (g\circ f)^! \to f^! \circ g^!$ is an isomorphism if one of the following holds:*

(1) f is of finite tor-dimension (e.g. flat)
(2) g is proper
(3) $g \circ f$ is proper
(4) We restrict to the subcategory $D^+_{\mathrm{qc}}(X)$.

Lemma C.2. *Let $f\colon Y \to X$ be a 1-morphism in $(\mathrm{Sp}/S)_{Noeth,sep,fp}$. If f is proper, then there exists a natural isomorphism $f^\times \to f^!$.*

Lemma C.3. *Consider a 2-fibered square in $(\mathrm{Sp}/S)_{Noeth,sep,fp}$*

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

where u,v are flat. Then the base change morphism $\mathbf{L}u^* g^! \to f^! \mathbf{L}v^*$ is an isomorphism if one of the following conditions holds:

(1) f is of finite tor-dimension

(2) We restrict to the subcategory $D^+_{\mathrm{qc}}(Z)$.

More generally, consider a fibered square of Noetherian algebraic spaces

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

where v is flat and f is proper. Then the base change morphism $\mathbf{L}u^*g^\times E \to f^\times \mathbf{L}v^* E$ is an isomorphism if one of the following conditions holds:

(1) f is of finite tor-dimension and $E \in D_{\mathrm{qc}}(Z)$

(2) We restrict to the subcategory $D^+_{\mathrm{qc}}(Z)$.

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