RELATIVE FOURIER–MUKAI THEORY FOR NOETHERIAN SCHEMES

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Abstract. These are rough lecture notes based on a series given at the RGAS Summer School *Mirror Symmetry and Homological Methods in Algebraic Geometry* in Zaragoza, Spain, from June 8 to 12, 2026.

Contents

1. Introduction

Integral transforms are exact functors between derived categories of quasi-coherent sheaves on schemes. They are determined by a kernel on the fiber product. These were introduced by Mukai [Muk81] in the study of equivalences between derived categories of abelian varieties.

When an integral transform is an equivalence, the corresponding schemes are called *Fourier–Mukai partners over the base.* In some sense, Fourier–Mukai equivalences allows one to think of a variety as a 'moduli space of coherent sheaves' on another [BBH09]. In this way, they provide a bridge between geometry and the homological properties of coherent sheaves.

Integral transforms also arise naturally in the study of *fibrations.* A natural question is whether the singularities of a fibration are preserved under derived equivalence. They furthermore play a central role in reconstruction theorems of Bondal–Orlov type (see e.g. [AB95]). In some settings, including curves and (anti-)Fano varieties, derived equivalence forces the varieties to be isomorphic [Spe23, Bal11, Bal09, LM14].

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Initially, integral transforms were studied primarily on smooth varieties. In this setting, bounded pseudocoherent complexes coincide with perfect complexes. Moreover, perfect complexes are precisely the compact objects of the derived category of quasi-coherent sheaves. Categorically speaking, 'compact objects are as necessary to this subject as air is to breathing' [Nee01, pg. 3].

This is illustrated by the work of [AL12]. Loc. cit. showed that the (co)unit of an adjoint pair of integral transforms between separated schemes of finite type over a field can be expressed in a particularly convenient form. Namely, it is induced by a morphism of complexes whenever the kernels of the integral transforms are perfect complexes [AL12].

Why is this useful? Suppose that we want to determine whether an integral transform is an equivalence over a given field. One may pass to an algebraic closure and study the base changed transform. The key point is that (co)unit be an isomorphism is equivalent to the associated morphism of complexes being an isomorphism. Consequently, whether an integral transform between smooth varieties is an equivalence becomes a geometric property: it can be detected after any field extension and therefore both ascends and descends along field extensions. See [Orl02, Lemma 2.12].

The situation is more subtle for singular varieties. Fourier–Mukai equivalences between singular varieties need not arise from kernels being perfect complexes [DLM25, Proposition 4.6]. Moreover, the arguments of [AL12] do not appear to extend directly to kernels that are not perfect, and hence are not compact objects. Nevertheless, one may still ask whether the behavior of an integral transform on fibers determines its behavior globally.

For schemes of finite type over an algebraically closed field, this question has an affirmative answer. Specifically, [HCS07, HCS09] showed that full faithfulness and equivalence can be detected on closed fibers. See [HCS09, Proposition 2.15].

This serves as the starting point for these notes. In particular, we study the behavior of integral transforms inducing functors on categories of perfect complexes and bounded pseudocoherent complexes, their behavior under various base changes (e.g. faithfully flat covers and special fibers), and situations in which the base scheme is not a field. Additionally, we that the locus of points where equivalences or fully faithfuless occurs for fibers forms an open subset on the base.

These notes draw on the recent work of [GLMP25, DLM25], the work in preparation [GLP26], and the references therein. In particular, the philosophy here is heavily influenced by the ideas and techniques developed in [AJS23, HCS09, HCS07, Bal09, Riz17].

2. PRELIMINARIES

2.1. **Generation.** We discuss a form of generation for triangulated categories. See [BV03] for details. Let $\mathcal{T}$ be a triangulated category with shift functor $[1]\colon \mathcal{T} \to \mathcal{T}$. Consider a subcategory $S \subseteq \mathcal{T}$. A triangulated subcategory of $\mathcal{T}$ is called **thick** if it is closed under direct summands. Denote by $\langle S \rangle$ the smallest thick subcategory of $\mathcal{T}$ containing S. If S consists of a single object G, we write $\langle S \rangle := \langle G \rangle$. Set $\operatorname{add}(S)$ to be the smallest strictly full subcategory of $\mathcal{T}$ containing S that is closed under shifts, finite coproducts, and direct summands. Inductively, let $\langle S \rangle_0$ consist of all zero objects in $\mathcal{T}$, $\langle S \rangle_1 := \operatorname{add}(S)$, and

$$\langle S \rangle_n := \operatorname{add}\{\operatorname{cone}(\phi) \mid \phi \in \operatorname{Hom}_{\mathcal{T}}(\langle S \rangle_{n-1}, \langle S \rangle_1)\}.$$

It can be checked that $\langle S \rangle = \cup_{n=0}^{\infty} \langle S \rangle_n$. We say E is **finitely built by** S if $E \in \langle S \rangle$.

Example 2.1. Let $X = \operatorname{Spec}(R)$ where R is a regular ring. Then $\langle \mathcal{O}_X \rangle = D^b_{\mathrm{coh}}(X)$. More generally, let X be an affine Noetherian scheme and $Z \subseteq X$ be closed. If $P \in \operatorname{Perf}(X)$ satisfies $\operatorname{supp}(P) = Z$, then $\langle P \rangle = \operatorname{Perf}(X) \cap D^b_{\mathrm{coh},Z}(X)$. See [Nee92, Lemma 1.2].

If $\mathcal{T}$ admits small coproducts, then the collection of compact objects in $\mathcal{T}$ will be denoted by $\mathcal{T}^c$. These form a triangulated subcategory of $\mathcal{T}$. We say that $\mathcal{T}$ is **compactly generated** if it coincides with the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{T}^c$ and closed under small coproducts. Equivalently, $\mathcal{T}$ is compactly generated if, for any $E \in \mathcal{T}$ satisfying $\operatorname{Hom}(P,E) = 0$ for all $P \in \mathcal{T}^c$, one has $E \cong 0$ [SS03, Lemma 2.2.1]. Note that classical generators for $\mathcal{T}^c$ coincide with compact generators for $\mathcal{T}$ [Sta26, Tag 09SR].

Example 2.2. Let X be a quasi-compact quasi-separated scheme. Then $D_{\mathrm{qc}}(X)^c = \operatorname{Perf}(X)$, and moreover, $\operatorname{Perf}(X)$ admits a classical generator. See [BV03, Theorem 3.1.1]. More generally, let $Z \subseteq X$ be closed. By [Rou08, Theorem 6.8], $D_{\mathrm{qc},Z}(X)$ is compactly generated by a single object. Hence, $\operatorname{Perf}(X) \cap D_{\mathrm{qc},Z}(X)$ admits a classical generator.

2.2. ***t*-structures.** We discuss t-structures on a triangulated category $\mathcal{T}$ and recall material from [KV88, BBDG18]. A pair of strictly full subcategories $\tau = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of $\mathcal{T}$ is a ***t*-structure** if:

- $\operatorname{Hom}(A,B) = 0$ for all $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 0}[-1]$,
- $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}[-1] \subseteq \mathcal{T}^{\geq 0}$,
- for every $E \in \mathcal{T}$, there is a distinguished triangle
$$\tau^{\leq 0}E \to E \to \tau^{\geq 1}E \to (\tau^{\leq 0}E)[1]$$
with $\tau^{\leq 0}E \in \mathcal{T}^{\leq 0}$ and $\tau^{\geq 1}E \in \mathcal{T}^{\geq 0}[-1]$.

The above distinguished triangle is unique up to unique isomorphism, and it is called the **truncation triangle** of E with respect to τ. Given $n \in \mathbb{Z}$, the pair $(\mathcal{T}^{\leq n}, \mathcal{T}^{\geq n})$ is also a t-structure on $\mathcal{T}$ where $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$. A pair of t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ are **equivalent** if there exists an $n \geq 0$ such that $\mathcal{T}_2^{\leq -n} \subseteq \mathcal{T}_1^{\leq 0} \subseteq \mathcal{T}_2^{\leq n}$.

Example 2.3. Assume that $\mathcal{T}$ admits small coproducts. Let $\mathcal{A}$ be a full subcategory of $\mathcal{T}^c$ closed under positive shifts. Denote by $\operatorname{Coprod}(\mathcal{A})$ the smallest strictly full subcategory of $\mathcal{T}$ that contains $\mathcal{A}$ which is closed under extensions and small coproducts. By [CHNS24, Theorem 2.3.3 & Remark 2.3.4] (which generalizes [ATLS03, Theorem A.1 & Proposition A.2]), this construction defines an aisle in $\mathcal{T}$. We call the associated t-structure $\tau_{\mathcal{A}}$ the ***t*-structure compactly generated by** $\mathcal{A}$. If $\mathcal{A} = \{G[i] \mid i \geq 0\}$ for some compact object G; we denote the corresponding compactly generated t-structure by τ_G. If $\mathcal{T}$ is compactly generated by a single object G, we define the **preferred equivalence class** to be the equivalence class of t-structures containing the t-structure compactly generated by G.

Let $F \colon \mathcal{T}_1 \to \mathcal{T}_2$ be an exact functor between triangulated categories equipped with t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$. We say that F is **right *t*-exact** if $F(\mathcal{T}_1^{\leq 0}) \subseteq \mathcal{T}_2^{\leq 0}$, and **left *t*-exact** if $F(\mathcal{T}_1^{\geq 0}) \subseteq \mathcal{T}_2^{\geq 0}$. If both conditions hold, then F is ***t*-exact**. We say τ is **nondegenerate** if $\cap_{n\in\mathbb{Z}}\mathcal{T}^{\geq n} = \cap_{n\in\mathbb{Z}}\mathcal{T}^{\leq n} = \operatorname{add}(0)$ (i.e. consists of only zero objects). Moreover, τ is called **bounded** if for every $E \in \mathcal{T}$ there exists an $n \geq 0$ such that $E[n] \in \mathcal{T}^{\leq 0}$ and $E[-n] \in \mathcal{T}^{\geq 0}$.

Lemma 2.4. *Let $\mathcal{T}$ be a triangulated category. A bounded t-structure $\tau = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on $\mathcal{T}$ is nondegenerate.*

Proof. Let $E \in \cap_{n\in\mathbb{Z}} \mathcal{T}^{\geq n}$. Assume E is not the zero object. Since τ is bounded, there is an $N \geq 0$ such that $E[N] \in \mathcal{T}^{\leq 0}$ and $E[-N] \in \mathcal{T}^{\geq 0}$. This means $E \in \mathcal{T}^{\leq N} \cap \mathcal{T}^{\geq -N}$. However, $E \in \mathcal{T}^{\geq N+1} = \mathcal{T}^{\geq N}[-1]$, and so, $E \in \mathcal{T}^{\leq N} \cap \mathcal{T}^{\geq N+1}$. Since $\operatorname{Hom}(A, B) = 0$ for all $A \in \mathcal{T}^{\leq N}$ and $B \in \mathcal{T}^{\geq N}[-1]$, we have a contradiction. A similar argument shows $\cap_{n\in\mathbb{Z}} \mathcal{T}^{\leq n} = \operatorname{add}(0)$. □

2.3. **Geometric constructions**. Let X be a quasi-compact quasi-separated scheme. The underlying topological space of X is given by equivalence classes of morphisms from fields to the stack (see [Sta26, Tag 04XE]). We denote it by $|X|$. By [Sta26, Tags 03BV & 03BU], the points of X (viewed as a topological space) are in one-to-one correspondence with those of $|X|$. This view is not standard.

2.3.1. *Categories.* $\operatorname{Mod}(X)$ is the Grothendieck abelian category of sheaves of $\mathcal{O}_X$-modules on X. $\operatorname{Qcoh}(X)$ is the full subcategory of $\operatorname{Mod}(X)$ consisting of quasi-coherent sheaves. $D(X) := D(\operatorname{Mod}(X))$ is the derived category of $\operatorname{Mod}(X)$. $D_{\mathrm{qc}}(X)$ is the full subcategory of $D(X)$ consisting of complexes with quasi-coherent cohomology sheaves. $\operatorname{Perf}(X)$ is the full subcategory of perfect complexes in $D_{\mathrm{qc}}(X)$. If X is Noetherian, then $\operatorname{coh}(X)$ is the full subcategory of $\operatorname{Mod}(X)$ consisting of coherent sheaves and $D^b_{\mathrm{coh}}(X)$ denotes the full subcategory of $D(X)$ consisting of bounded pseudocoherent complexes.

2.3.2. *Perfect complexes.* A complex is **strictly perfect** if it is a bounded complex whose terms are direct summands of finite free modules. A complex is **perfect** if it is locally strictly perfect. The compacts of $D_{\mathrm{qc}}(X)$ coincide with the perfect complexes.

Remark 2.5. We give a reminder of a useful topological fact. By [Sta26, Tag 0GW2], a quasi-separated scheme is decent. Moreover, [Sta26, Tag 0GW7] shows that a decent scheme has a Kolmogorov underlying topological space. Also, from [Sta26, Tag 005E], any nonempty quasi-compact Kolmogorov topological space contains a closed point. Hence, any $p \in |X|$ is the generalization of a closed point $p' \in |X|$.

2.3.3. *Internal homs.* Let $E, G \in D(X)$. There exists the derived tensor product $E \otimes^{\mathbf{L}} G \in D(X)$ and derived sheaf Hom $\mathbb{R}\mathcal{H}om(E, G) \in D(X)$. Here, $(-) \otimes^{\mathbf{L}} E$ is left adjoint to $\mathbb{R}\mathcal{H}om(E, -)$ on $D(X)$. If $E, G \in D_{\mathrm{qc}}(X)$, then $E \otimes^{\mathbf{L}} G \in D_{\mathrm{qc}}(X)$. However, this need not be the case for $\mathbb{R}\mathcal{H}om(E, G)$ despite the formation of $\mathbb{R}\mathcal{H}om(E, -)$ being smooth local.

By [HNR19, Theorem B.1], the category $D_{\mathrm{qc}}(X)$ is well generated. Also, the endofunctor $(-)\otimes^{\mathbf{L}} E$ on $D_{\mathrm{qc}}(X)$ preserves small coproducts. Hence, [Nee01, Theorem 8.4.4] implies that the endofunctor admits a right adjoint $\mathbf{R}\mathcal{H}om(E, -)$ on $D_{\mathrm{qc}}(X)$. Denote by $i \colon D_{\mathrm{qc}}(X) \to D(X)$ for the natural inclusion. It admits a right adjoint $Q \colon D(X) \to D_{\mathrm{qc}}(X)$ by reasoning above because i preserves small coproducts.

The following are well-known (see e.g. [HR17, §1.2] or [Nee23, §2]) but we add them for convenience.

Lemma 2.6. *Let X be an scheme. The natural morphism*

$$\mathbb{R}\mathcal{H}om(M, L) \to \mathbb{R}\mathcal{H}om(K \otimes^{\mathbf{L}} M, L)$$

is an isomorphism if any of the following conditions hold:

(1) *K is perfect*

(2) *L is perfect*
(3) *M is pseudocoherent, $L \in D^+(X)$, and K has finite tor-dimension.*

Proof. This is [Sta26, Tag 0ATN]. ☐

Lemma 2.7. *Let X be an scheme. For any $E, L \in D_{\mathrm{qc}}(X)$, there exists a natural isomorphism*

$$\mathbf{R}\mathcal{H}om(E,L) \to Q(\mathbb{R}\mathcal{H}om(i(E), i(L))).$$

In particular, if $\mathbb{R}\mathcal{H}om(i(E), i(L))$ has quasi-coherent cohomology, then

$$\mathbf{R}\mathcal{H}om(i(E), i(L)) \cong \mathbb{R}\mathcal{H}om(i(E), i(L)).$$

Proof. Recall that we have adjunctions,

$$\begin{tikzcd} D_{\mathrm{qc}}(X) \arrow[r, bend left, "i"] & D(X) \arrow[l, bend left, "Q"] \arrow[r, bend left, "((-)\otimes^{\mathbf{L}} i(E))"] & D(X). \arrow[l, bend left, "\mathbb{R}\mathcal{H}om(i(E),-)"] \end{tikzcd}$$

This gives us the composition of adjunctions,

$$\begin{tikzcd} D_{\mathrm{qc}}(X) \arrow[r, bend left, "((-)\otimes^{\mathbf{L}} i(E))\circ i"] & D(X). \arrow[l, bend left, "Q(\mathbb{R}\mathcal{H}om(i(E),-))"] \end{tikzcd}$$

Note that $((-) \otimes^{\mathbf{L}} i(E)) \circ i$ induces an endofunctor on $D_{\mathrm{qc}}(X)$ (e.g. use that i is monoidal) whose right adjoint is $\mathbf{R}\mathcal{H}om(E, -)$. Then restriction gives an adjunction

$$\begin{tikzcd} D_{\mathrm{qc}}(X) \arrow[r, bend left, "((-)\otimes^{\mathbf{L}} i(E))\circ i"] & D_{\mathrm{qc}}(X). \arrow[l, bend left, "Q(\mathbb{R}\mathcal{H}om(i(E),-))"] \end{tikzcd}$$

Thus, the desired claim follows from uniqueness of adjoints. ☐

Lemma 2.8. *Let $f : Y \to X$ be a morphism of schemes. Choose $E, G \in D(X)$. Assume one of the following holds:*

(1) *E is perfect*
(2) *E is pseudocoherent and both $G, \mathbf{L}f^*G$ are locally bounded below.*

Then there is an isomorphism

$$\mathbf{L}f^* \mathbf{R}\mathcal{H}om(E, G) \to \mathbf{R}\mathcal{H}om(\mathbf{L}f^*E, \mathbf{L}f^*G).$$

Proof. See e.g. [GLMP25, Lemma 2.2]. ☐

2.3.4. *Quasi-affine diagonal.* We give a characterization of quasi-affine diagonal. It will be used freely throughout. A useful fact is the quasi-compact quasi-separated schemes have quasi-affine diagonal [Sta26, Tags 02X4, 04XS, & 0ABS]. In fact, this is the only take away for this subsection, and the rest may be ignored.

Lemma 2.9. *Consider a commutative diagram of schemes*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[dr, "g"'] & Y \arrow[d, "f"] \\ & X. \end{tikzcd}$$

If both g and Δ_f are quasi-affine, then h is quasi-affine.

Proof. There exists a fibered square

$$\begin{tikzcd}
Z \times_X Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "f"] \\
Z \arrow[r, "g"'] & X.
\end{tikzcd}$$

By base change, p_2 is quasi-affine (see e.g. [Sta26, Tag's 0302, 045C, 03WO, & 0423]). Moreover, the morphism $(1, f)\colon Z \to Z \times_X Y$ is the base change of the diagonal $\Delta_f\colon Y \to Y \times_X Y$ by the morphism $Z \times_X Y \to Y \times_X Y$ [Sta26, Tag 003O]. Again, by base change, $(1, f)\colon Z \to Z \times_X Y$ is quasi-affine. Since $f = p_2 \circ (1, f)$ is the composition of quasi-affine morphisms, it must be itself quasi-affine [Sta26, Tag's 0301 & 045B]. ☐

Proposition 2.10. *Let X be a quasi-compact scheme. Then the following are equivalent:*

1. *X has quasi-affine diagonal*
2. *there exists a quasi-affine smooth surjective morphism to X from a quasi-affine scheme*
3. *every morphism to X from a quasi-affine scheme is quasi-affine.*

Proof. First, let X have quasi-affine diagonal. Choose a smooth surjective morphism $s\colon U \to S$ from a quasi-affine scheme. By Lemma 2.9, s is quasi-affine.

Next, let $s\colon U \to X$ be a quasi-affine smooth surjective morphism from a quasi-affine scheme. Choose a morphism $t\colon V \to X$ from a quasi-affine scheme. Consider the projection morphisms $t'\colon V \times_X U \to U$ and $s'\colon V \times_X U \to V$. By base change, s' is quasi-affine. In particular, $V \times_X U$ is quasi-affine. Since t' is a morphism from a quasi-affine scheme to a quasi-separated scheme, it follows that t must be quasi-affine [Sta26, Tag 054G]. As s is smooth and surjective, we see that t is quasi-affine.

Lastly, assume every morphism to X from a quasi-affine scheme is quasi-affine. There is a fibered square

$$\begin{tikzcd}
U \times_X U \arrow[r, "s_2"] \arrow[d, "s_1"'] & U \arrow[d, "s"] \\
U \arrow[r, "s"'] & X.
\end{tikzcd}$$

By base change, each s_i is quasi-affine. Hence, $U \times_X U$ is a quasi-affine scheme. Consider the commutative diagram

$$\begin{tikzcd}
U \times_{\mathbb{Z}} U \arrow[r] \arrow[d, "s_1"'] & X \times_{\mathbb{Z}} U \arrow[r] \arrow[d] & U \arrow[d, "s"] \\
U \times_{\mathbb{Z}} X \arrow[r, "s_1'"'] \arrow[d] & X \times_{\mathbb{Z}} X \arrow[r] \arrow[d] & X \arrow[d] \\
U \arrow[r, "s"'] & X \arrow[r] & \mathrm{Spec}(\mathbb{Z}).
\end{tikzcd}$$

By base change, s_1 and s_1' are smooth and surjective. Denote by $s' = s_1' \circ s_1$. Appealing to [Sta26, Tag 04Z1], there is a fibered square

$$\begin{tikzcd}
U \times_X U \arrow[r, "t"] \arrow[d, "t'"'] & U \times_{\mathbb{Z}} U \arrow[d, "s'"] \\
X \arrow[r, "\Delta_X"'] & X \times_{\mathbb{Z}} X.
\end{tikzcd}$$

The source and target of t are quasi-affine schemes, and so, t is quasi-affine. Since s' is smooth and surjective, [Sta26, Tag's 04XD & Tag 04XS] shows that Δ_X is quasi-affine. □

2.3.5. *Support.* For any $M \in \mathrm{Qcoh}(X)$, define $\mathrm{supp}(M)$ in the usual way, see e.g. [Sta26, Tag 056H]. For $E \in D_{\mathrm{qc}}(X)$, set

$$\mathrm{supp}(E) := \bigcup_{j \in \mathbb{Z}} \mathrm{supp}\left(\mathcal{H}^j(E)\right) \subseteq |X|.$$

Given a closed subset $Z \subseteq |X|$, we say E is **supported on** Z if $\mathrm{supp}(E) \subseteq Z$.

Lemma 2.11. *Let X be a scheme and $p \in |X|$. If $t\colon \mathrm{Spec}(k) \to X$ is a morphism from a field such that $t(\mathrm{Spec}(k)) = p$, then t represents p (see* [Sta26, Tag 04XE]*).*

Proof. Let $i\colon Z_p := \mathrm{Spec}(\kappa(p)) \to X$ be the natural morphism. Choose any $h\colon \mathrm{Spec}(\ell) \to X$ that represents p. By [Sta26, Tag 0DTH], there exists a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r, "t'"] \arrow[dr, "t"'] & Z_p \arrow[d, "i"] \\
& X.
\end{tikzcd}$$

Moreover, from [Sta26, Tag 06MW], there exists a commutative diagram

$$\begin{tikzcd}
Z_p \arrow[d, "i"'] & \mathrm{Spec}(\ell) \arrow[l, "h'"'] \arrow[dl, "h"] \\
X. &
\end{tikzcd}$$

Consider the fiber product

$$\begin{tikzcd}
\mathrm{Spec}(k) \times_{Z_p} \mathrm{Spec}(\ell) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathrm{Spec}(\ell) \arrow[d, "h'"] \\
\mathrm{Spec}(k) \arrow[r, "t'"'] & Z_p.
\end{tikzcd}$$

Since $|Z_p|$ is a singleton [Sta26, Tag 06MT], t' is surjective. Hence, by base change, p_2 is surjective. Thus, $\mathrm{Spec}(k) \times_{Z_p} \mathrm{Spec}(\ell)$ is nonempty. Here $\mathrm{Spec}(k) \times_{Z_p} \mathrm{Spec}(\ell)$ is an scheme. Choose an étale surjective morphism $s\colon U \to \mathrm{Spec}(k) \times_{Z_p} \mathrm{Spec}(\ell)$. Fix some $p' \in |\mathrm{Spec}(k) \times_{Z_p} \mathrm{Spec}(\ell)|$. Consider any $g\colon \mathrm{Spec}(K) \to \mathrm{Spec}(k) \times_{Z_p} \mathrm{Spec}(\ell)$ that represents p'. Then we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r, "p_2\circ g"] \arrow[d, "p_1\circ g"'] & \operatorname{Spec}(\ell) \arrow[d, "h"] \\
\operatorname{Spec}(k) \arrow[r, "t"'] & X.
\end{tikzcd}$$

This completes the proof. □

Lemma 2.12. *Let X be a scheme. Then* $\operatorname{supp}(E)$ *is closed in* $|X|$ *for any* $E \in \operatorname{Qcoh}(X)$ *that is of finite type.*

Proof. Note that $\operatorname{supp}(E) = s(\operatorname{supp}(s^*E))$. There is nothing to check if $E \cong 0$. We can impose E be nonzero. It suffices to prove that $|X| \setminus \operatorname{supp}(E)$ is open in $|X|$. In fact, we show that

$$s(U) \setminus s(\operatorname{supp}(s^*E)) = |X| \setminus \operatorname{supp}(E) = s(U \setminus \operatorname{supp}(s^*E)).$$

This is equivalent to checking $s(U \setminus \operatorname{supp}(s^*E)) \cap s(\operatorname{supp}(s^*E)) = \emptyset$. Assume the contrary. Let $p \in s(U \setminus \operatorname{supp}(s^*E)) \cap s(\operatorname{supp}(s^*E))$. There exists $p_1 \in s(U \setminus \operatorname{supp}(s^*E))$ and $p_2 \in s(\operatorname{supp}(s^*E))$ such that $s(p_i) = p$. Now, by Lemma 2.11, $\operatorname{Spec}(\kappa(p_i)) \xrightarrow{j_i} U \xrightarrow{s} X$ belong to the same equivalence class. Hence, there exists a field k and commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[r, "q_2"] \arrow[d, "q_1"'] & \operatorname{Spec}(\kappa(p_2)) \arrow[d, "s\circ j_2"] \\
\operatorname{Spec}(\kappa(p_1)) \arrow[r, "s\circ j_1"'] & X
\end{tikzcd}$$

where q_i are morphisms. By [Sta26, Tag 056J], the choice of p_1 says $(s \circ j_1)^*E \cong 0$ and the choice of p_2 says $(s \circ j_2)^*E \ncong 0$. However, the diagram implies

$$0 \cong (s \circ j_1 \circ q_1)^*E \cong (s \circ j_2 \circ q_2)^*E \ncong 0.$$

Therefore, we have a contradiction. □

Corollary 2.13. *Let X be an scheme. Then* $\operatorname{supp}(E)$ *is closed for any* $E \in D_{\mathrm{qc}}(X)$ *whose cohomology is of finite type.*

Proof. This follows from Lemma 2.12 and the string of equalities:

$$\begin{aligned}
\operatorname{supp}(E) &= \bigcup_n \operatorname{supp}(\mathcal{H}^n(E)) \\
&= \bigcup_n s(\operatorname{supp}(s^*\mathcal{H}^n(E))) \\
&= \bigcup_n s(\operatorname{supp}(\mathcal{H}^n(s^*E))) \\
&= \bigcup_n s(\operatorname{supp}(\mathcal{H}^n(\mathbf{L}s^*E))) \\
&= s(\bigcup_n \operatorname{supp}(\mathcal{H}^n(\mathbf{L}s^*E))) \\
&= s(\operatorname{supp}(\mathbf{L}s^*E)).
\end{aligned}$$

□

2.3.6. *Singularity category.* The **singularity category** of X, denoted $D_{\mathrm{sg}}(X)$, is defined as the Verdier localization of $D^b_{\mathrm{coh}}(X)$ by $\mathrm{Perf}(X)$. This category was first introduced in the algebra setting [Buc21], and later rediscovered in the geometric setting by [Orl04]. One can check that X is regular if, and only if, $D_{\mathrm{sg}}(X)$ is trivial. Hueristically, the structure of $D_{\mathrm{sg}}(X)$ 'reflects' the singularities of X.

We say two Noetherian schemes are **derived equivalent** if their bounded derived categories of coherent sheaves are triangle equivalent. This is equivalent to detecting triangulated equivalences between their derived category of quasi-coherent sheaves and/or category of perfect complexes (see [CNS25a, Corollary 5.4]).

There has been recent attention to detecting triangulated equivalences between singularity categories, which leads to the notion: We say two Noetherian schemes X, Y are **singular equivalent** if there is a triangulated equivalence $D_{\mathrm{sg}}(X) \to D_{\mathrm{sg}}(Y)$. This has been detected in various cases. See e.g. [Kal21, Knö87, Mat19, MT17]. It follows from [CNS25a, Corollary 5.8] that derived equivalences imply singular equivalences in our setting.

2.3.7. *Integral transforms.* Let $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ be morphisms of finite type with S a Noetherian scheme. Consider the fibered square:

$$\begin{tikzcd}
Y_1 \times_S Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_2 \arrow[d, "f_2"] \\
Y_1 \arrow[r, "f_1"'] & S
\end{tikzcd}$$

The **integral S-transform** associated to an object K in $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$ is the functor Φ_K from $D_{\mathrm{qc}}(Y_1)$ to $D_{\mathrm{qc}}(Y_2)$ given by $\mathbf{R}p_{2,*}(\mathbf{L}p_1^*(-) \otimes^{\mathbf{L}} K)$. We will drop the hyphen 'S-' if it is clear from context. One says Y_1 and Y_2 are **Fourier–Mukai S-partners** if there is such a K for which Φ_K yields an equivalence $D^b_{\mathrm{coh}}(Y_1) \to D^b_{\mathrm{coh}}(Y_2)$.

This notion was first introduced in [Muk81] for varieties. These functors have been studied for when they restrict, induce equivalences, or admit adjoints on other subcategories (e.g. $D^b_{\mathrm{coh}}(-)$ or $\mathrm{Perf}(-)$). See e.g. [Sta26, Tag 0FYP].

A complete list of references on integral transforms is not reasonable, but we highlight a few for the readers interest: [Orl97, Bal09, HCS07, HCS09, Riz17, RVdBN19, RVdB15]. These have also been studied in the setting of algebraic stacks (see [BS20, §3], [HP24], and more general theory in [GLP26]). See [Huy06, Muk81] for further background.

We record a few cases of interest where singular equivalences can be detected through integral transforms.

Example 2.14.

(1) Let X be a quasi-projective variety over a field. Suppose $j \colon U \to X$ is an open immersion such that the singular locus of X is contained in U. Then $j^* \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(U)$ induces a triangulated equivalence $\breve{j}^* \colon D_{\mathrm{sg}}(X) \to D_{\mathrm{sg}}(U)$. See [Che10, Corollary 2.3] or [Orl04, Proposition 1.14]. Observe that the derived pullback $j^* \colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(U)$ can be realized as the integral transform associated to the graph of j.

(2) Let Y be a smooth quasi-projective variety over a field k and $f \colon Y \to \mathbb{A}^1_k$ be a nonzero morphism. Define $g = f + xy \colon Y \times_k \mathbb{A}^2_k \to \mathbb{A}^1_k$, $Z_f := f^{-1}(\{0\})$, $Z_g :=$

$g^{-1}(\{0\})$, and $W := Z_f \times_k \{0\} \times_k \mathbb{A}^1_k$ (in $Y \times_k \mathbb{A}^2_k$). Denote by $i\colon W \to Z_g$ the inclusion and $p\colon W \to Z_f$ the flat projection. Then $\mathbf{R}i_* p^*\colon D_{\mathrm{sg}}(Z_f) \to D_{\mathrm{sg}}(Z_g)$ is a triangulated equivalence, see [Orl04, Theorem 2.1]. This can be expressed as a composition of integral transforms.

Remark 2.15. There are exact functors between D^b_{coh} which do not arise from integral transform. These are difficult to find. Initially, this was accomplished in [RVdBN19, Theorem 1.4], which has been further refined by [RRVdB22, Thoerem 1.3] and [Kün24]. Recently, [CO26] has found more examples.

3. REVISITING DEFINITIONS

We clarify that all notions of 'relatively perfect' complexes for morphisms of schemes are equivalent. See Proposition 3.3. In fact, this result seems to be missing in the literature, e.g. see comments in [AJS23, §1.4] and below [HCS07, Definition 1.3].

Lemma 3.1. *Let $\mathcal{T}$ be a triangulated category equipped with a bounded t-structure τ. Denote by H^n_τ the n-th cohomology functor with respect to τ. Then any object E in $\mathcal{T}$ is finitely built by $\oplus_{n\in\mathbb{Z}} H^n_\tau(E)$.*

Proof. Fix $E \in \mathcal{T}$. We prove the claim by induction using the truncation triangles

$$\tau^{\leq k-1}E \to \tau^{\leq k}E \to H^k_\tau(E)[-k] \to (\tau^{\leq k-1}E)[1].$$

By Lemma 2.4, τ is nondegenerate. This implies that the zero objects coincide with those satisfying s-th cohomology vanishes for all $s \in \mathbb{Z}$. Hence, $\langle \{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_0$ consists of only zero objects. Thus, we can impose $E \ncong 0$. Set $k_1 = \min\{n \in \mathbb{Z} : H^n_\tau(E) \ncong 0\}$. Then $\tau^{\leq k_1-1}E \cong 0$, and so, $\tau^{\leq k_1}E \cong H^{k_1}_\tau(E)[-k_1]$. Assume by induction that there exists an $N \geq 0$ such that for all $0 \leq s \leq N$ we have $\tau^{\leq k_1+s}E \in \langle \{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_{s+1}$. Consider the truncation triangle

$$\begin{aligned}\tau^{\leq k_1+(N+1)-1}E &\to \tau^{\leq k_1+(N+1)}E \\ &\to H^{k_1+(N+1)}_\tau(E)[-(k_1+N+1)] \to (\tau^{\leq k_1+(N+1)-1}E)[1].\end{aligned}$$

By the inductive hypothesis,

$$\tau^{\leq k_1+(N+1)-1}E \in \langle \{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_{N+1},$$

which implies that $\tau^{\leq k_1+(N+1)}E \in \langle \{H^n_\tau(E)\}_{n\in\mathbb{Z}}\rangle_{N+2}$. Since τ is a bounded t-structure, $H^n_\tau(E) \cong 0$ for $|n| \gg 0$, and hence $E \cong \tau^{\leq k_1+L}E$ for $L \gg 0$. This completes the proof. □

Lemma 3.2. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated schemes. For any $E \in D^b_{\mathrm{qc}}(Y)$, the following are equivalent:*

1. *$E \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{qc}}(X) \subseteq D^b_{\mathrm{qc}}(Y)$*
2. *there exists $[a,b] \subseteq \mathbb{Z}$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \cong 0$ for all $j \notin [a,b]$ and $M \in \operatorname{Qcoh}(X)$.*

Proof. First, we show (1) $\implies$ (2). Assume the contrary. Then for each $n \geq 1$ there exists an $M_n \in \operatorname{Qcoh}(Y)$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \ncong 0$ for some $j \notin [-n,n]$. Set $M := \oplus_{n\geq 1} M_n$. By hypothesis, $E \otimes^{\mathbf{L}} \mathbf{L}f^*M \in D^b_{\mathrm{qc}}(Y)$, and hence there exists $[a,b] \subseteq \mathbb{Z}$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \cong 0$ for all $j \notin [a,b]$. After shifting, if necessary, we can impose $a = 0$. Since $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M_n)$ is a direct summand of $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M)$ for each $n \geq 1$,

$\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M_n) \cong 0$ for all $j \notin [0,b]$ and $n \geq 1$. This leads to a contradiction. Indeed, for $n > b$ there exists $j \notin [0,n]$ with $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M_n) \ncong 0$.

Next, we prove that (2) $\implies$ (1). Let $B \in D^b_{\mathrm{qc}}(X)$. By Lemma 3.1, B is finitely built by its cohomology sheaves $\mathcal{H}^j(B)$. Moreover, the hypothesis implies $\mathbf{L}f^*\{\mathcal{H}^i(B)\}_{i\in\mathbb{Z}} \subseteq D^b_{\mathrm{qc}}(Y)$. Since $B \in \langle\{\mathcal{H}^i(B)\}\rangle$, it follows that

$$E \otimes^{\mathbf{L}} \mathbf{L}f^*B \in \langle\{E \otimes^{\mathbf{L}} \mathbf{L}f^*\mathcal{H}^i(B)\}\rangle \subseteq D^b_{\mathrm{qc}}(Y).$$

This completes the proof. □

Proposition 3.3. *Let $f\colon Y \to X$ be a morphism of quasi-compact quasi-separated schemes. For any $E \in D^b_{\mathrm{qc}}(Y)$, the following are equivalent:*

1. $E \otimes^{\mathbf{L}} \mathbf{L}f^*D^b_{\mathrm{qc}}(X) \subseteq D^b_{\mathrm{qc}}(Y)$
2. *E has finite tor-dimension as an object of $D(f^{-1}\mathcal{O}_X)$.*

Proof. First, we show (1) $\implies$ (2). By Lemma 3.2, there exists $[a,b] \subseteq \mathbb{Z}$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \cong 0$ for all $j \notin [a,b]$ and $M \in \mathrm{Qcoh}(X)$. Following [LH09, pg. 78, Example 2.6.7], this condition is equivalent to E having finite flat f-amplitude in $[a,b]$ (see loc. cit. for definition). Moreover, this condition is equivalent to the requirement that for each $p \in Y$, E_p is isomorphic in $D(\mathcal{O}_{X,f(p)})$ to a complex of flat $\mathcal{O}_{X,f(p)}$-modules vanishing in degrees outside of $[a,b]$; see comments above [LH09, Eq. 2.7.6.1]. Applying [Ill71, Proposition 3.3], this stalk local condition is equivalent to E having finite tor-dimension in $[a,b]$ as an object of $D(f^{-1}\mathcal{O}_X)$ (see also [GW20, Proposition 21.169 & Lemma 21.171]). It follows that (1) $\implies$ (2).

Conversely, the same references show that (2) in this proof implies (2) of Lemma 3.2. Thus, (2) $\implies$ (1). □

4. Preservation of small objects

We identify necessary and sufficient conditions under which integral transforms preserve perfect complexes and/or objects with bounded and coherent cohomology.

Lemma 4.1. *Let Y_1 and Y_2 be schemes that are proper over an affine Noetherian scheme* $\mathrm{Spec}(R)$*. Then, for any exact functor $\Phi\colon \mathrm{Perf}(Y_1) \to \mathrm{Perf}(Y_2)$, there is a unique exact functor $\Phi'\colon D^b_{\mathrm{coh}}(Y_2) \to D^b_{\mathrm{coh}}(Y_1)$, with natural isomorphisms*

$$\mathrm{Hom}(\Phi(A), B) \cong \mathrm{Hom}(A, \Phi'(B))$$

for any A in $\mathrm{Perf}(Y_1)$ and B in $D^b_{\mathrm{coh}}(Y_2)$.

Proof. It follows, by [Nee21, Example 0.7], that $D^b_{\mathrm{coh}}(Y_i)$ is equivalent to the category of finite cohomological functors $\mathrm{Perf}(Y_i)^{\mathrm{op}} \to \mathrm{Mod}(R)$. Consider a finite cohomological functor $H\colon \mathrm{Perf}(Y_2)^{\mathrm{op}} \to \mathrm{Mod}(R)$. It is easy to see that $H \circ \Phi^{\mathrm{op}}\colon \mathrm{Perf}(Y_1)^{\mathrm{op}} \to \mathrm{Mod}(R)$ is a finite cohomological functor on $\mathrm{Perf}(Y_1)$, which gives us the desired functor $\Phi'\colon D^b_{\mathrm{coh}}(Y_2) \to D^b_{\mathrm{coh}}(Y_1)$. See [Nee21, Definition 0.1] for terminology. □

Proposition 4.2. *Let $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ be morphisms of finite type to a Noetherian scheme. Consider the fibered square:*

$$\begin{tikzcd}
Y_1 \times_S Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "f_2"] \\
Y_1 \arrow[r, "f_1"'] & S
\end{tikzcd}$$

Suppose K is an object in $D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Then $\Phi_K(\operatorname{Perf}(Y_1))$ is contained in $\operatorname{Perf}(Y_2)$ if, and only if, $\mathbf{R}p_{2,*}(K \otimes^{\mathbf{L}} \operatorname{Perf}(Y_1 \times_S Y_2))$ is contained in $\operatorname{Perf}(Y_2)$.

Proof. Observe the converse direction follows from the fact that $\mathbf{L}p_1^* \operatorname{Perf}(Y_1)$ is contained in $\operatorname{Perf}(Y_1 \times_S Y_2)$. So we only need to check the forward direction. Assume that $\Phi_K(\operatorname{Perf}(Y_1))$ is contained in $\operatorname{Perf}(Y_2)$. Let P_i be classical generators for $\operatorname{Perf}(Y_i)$. Then, by [BV03, Lemma 3.4.1] coupled with Example 2.2, one has that $\mathbf{L}p_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}P_2^*P_2$ is a classical generator for $\operatorname{Perf}(Y_1 \times_S Y_2)$. Our hypothesis says that $\Phi_K(P_1)$ is an object of $\operatorname{Perf}(Y_2)$. However, by projection formula, we know that

$$\Phi_K(P_1) \otimes^{\mathbf{L}} P_2 \cong \mathbf{R}p_{2,*}(K \otimes^{\mathbf{L}} \mathbf{L}p_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^*P_2).$$

Hence, as $(-) \otimes^{\mathbf{L}} P_2$ is an endofunctor on $\operatorname{Perf}(Y_2)$, one has

$$\mathbf{R}p_{2,*}(K \otimes^{\mathbf{L}} \mathbf{L}p_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^*P_2) \subseteq \operatorname{Perf}(Y_2).$$

Then, as $\mathbf{L}p_1^*P_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^*P_2$ is a classical generator for $\operatorname{Perf}(Y_1 \times_S Y_2)$, $\mathbf{R}p_{2,*}(K \otimes^{\mathbf{L}} \operatorname{Perf}(Y_1 \times_S Y_2))$ is contained in $\operatorname{Perf}(Y_2)$. This completes the proof. □

Lemma 4.3. *Let $f \colon Y \to X$ be a morphism between schemes which are proper over an affine Noetherian scheme. Then an object E in $D^-_{\mathrm{coh}}(Y)$ satisfies $\mathbf{R}f_*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y))$ being contained in $\operatorname{Perf}(X)$ if, and only if, $E \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{coh}}(X)$ is contained in $D^b_{\mathrm{coh}}(Y)$.*

Proof. First, we show the forward direction. Let E be an object of $D^-_{\mathrm{coh}}(Y)$. Assume $\mathbf{R}f_*(E \otimes^{\mathbf{L}} \operatorname{Perf}(Y))$ is contained in $\operatorname{Perf}(X)$. Then, by [Bal09, Lemma 3.7], we have $\mathbf{R}f_*(\mathbf{R}\mathcal{H}om(E, f^!\mathcal{O}_X) \otimes^{\mathbf{L}} P)$ is an object of $\operatorname{Perf}(X)$ for each P in $\operatorname{Perf}(Y)$. Observe that [Bal09, Lemma 3.10] tells us $\mathbf{R}f_*(\mathbf{R}\mathcal{H}om(E, f^!\mathcal{O}_X) \otimes^{\mathbf{L}} (-))$ is left adjoint to $E \otimes^{\mathbf{L}} \mathbf{L}f^*(-)$ as functors between $D_{\mathrm{qc}}(X)$ and $D_{\mathrm{qc}}(Y)$. Then, by Lemma 4.1, one has $E \otimes^{\mathbf{L}} \mathbf{L}f^*(-) \colon D^b_{\mathrm{coh}}(X) \to D_{\mathrm{qc}}(Y)$ must factor through $D^b_{\mathrm{coh}}(Y)$ as desired.

Next, we check the converse direction. Suppose $E \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{coh}}(X)$ is contained in $D^b_{\mathrm{coh}}(Y)$. Note that f is proper as a morphism between proper schemes over a scheme must itself be proper. This ensures that $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P)$ belongs to $D^b_{\mathrm{coh}}(X)$. Now, observe that for any P in $\operatorname{Perf}(Y)$, one has that $E \otimes^{\mathbf{L}} P \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{coh}}(X)$ is also contained in $D^b_{\mathrm{coh}}(Y)$. Then, by the projection formula, we see that $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P) \otimes^{\mathbf{L}} D^b_{\mathrm{coh}}(X)$ is also contained in $D^b_{\mathrm{coh}}(X)$ for all P in $\operatorname{Perf}(Y)$. As we already know that $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P)$ belongs to $D^b_{\mathrm{coh}}(X)$, it follows from [AJS23, Theorem 2.3.(3)] that $\mathbf{R}f_*(E \otimes^{\mathbf{L}} P)$ is contained in $\operatorname{Perf}(X)$ as desired, which completes the proof. □

Lemma 4.4. *Let $f_1\colon Y_1 \to S$ and $f_2\colon Y_2 \to S$ be proper morphisms to an affine Noetherian scheme. Consider the fibered square:*

$$\begin{tikzcd} Y_1 \times_S Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & S. \end{tikzcd}$$

Then the following are equivalent for any object E in $D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$:

(1) $\Phi_E(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$

(2) $\mathbf{R}p_{1,*}(E \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2)) \subseteq \mathrm{Perf}(Y_1)$.

Proof. First, we check (1) $\implies$ (2). Assume $\Phi_E(D^b_{\mathrm{coh}}(Y_1))$ is contained in $D^b_{\mathrm{coh}}(Y_2)$. It suffices, by Lemma 4.3, to show that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^* D^b_{\mathrm{coh}}(Y_1)$ is contained in $D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$.

Note, by [Sta26, Tag 09U7], that $\mathbf{L}p_1^* D^b_{\mathrm{coh}}(Y_1)$ is contained in $D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Hence, from [Sta26, Tag 09J3], one has $E \otimes^{\mathbf{L}} D^b_{\mathrm{coh}}(Y_1)$ is contained in $D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$. It suffices to check that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G$ has bounded cohomology for each G in $D^b_{\mathrm{coh}}(Y_1)$. This can be done by showing each such object belongs to $D^+_{\mathrm{qc}}(Y_1 \times_S Y_2)$.

Let P_i be a compact generator of $D_{\mathrm{qc}}(Y_i)$ for each i. Then, by [BV03, Lemma 3.4.1], one has $\mathbf{L}p_1^* P_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^* P_2$ is a compact generator for $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. There is a string of isomorphisms:

$$\begin{aligned}
&\mathrm{Ext}^n(\mathbf{L}p_1^* P_1 \otimes^{\mathbf{L}} \mathbf{L}p_2^* P_2, E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G) && \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^* P_2, \mathbf{R}\mathcal{H}om(\mathbf{L}p_1^* P_1, E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G)) && \text{([Sta26, Tag 08DH])} \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^* P_2, \mathbf{R}\mathcal{H}om(\mathbf{L}p_1^* P_1, \mathcal{O}_{Y_1 \times_S Y_2}) \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G) && \text{([Sta26, Tag 08DQ])} \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^* P_2, \mathbf{R}\mathcal{H}om(\mathbf{L}p_1^* P_1, \mathbf{L}p_1^* \mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G) && (\mathbf{L}p_1^* \mathcal{O}_{Y_1} = \mathcal{O}_{Y_1 \times_S Y_2}) \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^* P_2, \mathbf{L}p_1^*(\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1})) \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G) && \text{([GW23, Prop. 22.70])} \\
&\cong \mathrm{Ext}^n(\mathbf{L}p_2^* P_2, \mathbf{L}p_1^*(\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1}) \otimes G) \otimes^{\mathbf{L}} E) && \text{([Sta26, Tag 07A4])} \\
&\cong \mathrm{Ext}^n(P_2, \mathbf{R}p_{2,*}(\mathbf{L}p_1^*(\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1}) \otimes G) \otimes^{\mathbf{L}} E)) && \text{(Adjunction)} \\
&\cong \mathrm{Ext}^n(P_2, \Phi_E(\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1}) \otimes G)). &&
\end{aligned}$$

It follows, as $\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1})$ is perfect on Y_1, that $\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1}) \otimes G$ belongs to $D^b_{\mathrm{coh}}(Y_1)$. Our hypothesis tells us $\Phi_E(\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1}) \otimes G)$ belongs to $D^b_{\mathrm{coh}}(Y_2)$. Hence, from [Sta26, Tag 0GEQ], one has $\mathrm{Ext}^n(P_2, \Phi_E(\mathbf{R}\mathcal{H}om(P_1, \mathcal{O}_{Y_1}) \otimes G)) = 0$ for $0 \gg n$. Then, once more from [Sta26, Tag 0GEQ], we see that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^* G$ is an object of $D^+_{\mathrm{qc}}(Y_1 \times_S Y_2)$ as desired.

Next, we show (2) $\implies$ (1). It follows, by Lemma 4.3 coupled with the hypothesis, that $E \otimes^{\mathbf{L}} \mathbf{L}p_1^* D^b_{\mathrm{coh}}(Y_1)$ is contained in $D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Then, as p_2 is proper, we have $\mathbf{R}p_{2,*} D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ is contained in $D^b_{\mathrm{coh}}(Y_2)$. This completes the proof. ◻

Proposition 4.5. *Let $f_1\colon Y_1 \to S$ and $f_2\colon Y_2 \to S$ be proper morphisms to a Noetherian scheme. Consider the fibered square:*

$$\begin{tikzcd}
Y_1 \times_S Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "f_2"] \\
Y_1 \arrow[r, "f_1"'] & S.
\end{tikzcd}$$

Then the following are equivalent for any object K in $D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$:

(1) $\Phi_K(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$,

(2) $\mathbf{R}p_{1,*}(K \otimes^{\mathbf{L}} \operatorname{Perf}(Y_1 \times_S Y_2)) \subseteq \operatorname{Perf}(Y_1)$.

Proof. We only check (1) $\implies$ (2) as the (2) $\implies$ (1) direction can be argued in a similar fashion. Consider an affine open cover $U_1, \dots, U_n$ of S. This gives us an open cover Y'_{ij} for each Y_j. Denote by $s_i\colon U_i \to S$ the associated open immersion of each U_i in S.

There is, for each i, a commutative cube:

$$\begin{tikzcd}
Y'_{i1} \times_S Y'_{i2} \arrow[rr, "q_{i2}"] \arrow[dr, "q_{i1}"] \arrow[dd, "t_i"'] & & Y'_{i2} \arrow[dr, "g_{i2}"] \arrow[dd, dashed, "s_{i2}"' near start] & \\
& Y'_{i1} \arrow[rr, "g_{i1}" near end] \arrow[dd, "s_{i1}" near end] & & U_i \arrow[dd, "s_i"] \\
Y_1 \times_S Y_2 \arrow[rr, dashed, "p_2" near start] \arrow[dr, "p_1"] & & Y_2 \arrow[dr, "f_2"] & \\
& Y_1 \arrow[rr, "f_1"] & & S.
\end{tikzcd}$$

whose faces are fibered squares, each vertical edge is an open immersion, and every other edge is a proper morphism.

Observe, from the cube above, one has the following computation for each object E in $D_{\mathrm{qc}}(Y_1)$:

$$\begin{aligned}
s^*_{i2}\Phi_K(E) &\cong s^*_{i2}\mathbf{R}p_{2,*}(\mathbf{L}p^*_1 E \otimes^{\mathbf{L}} K) \\
&\cong \mathbf{R}q_{i2,*}t^*_i(\mathbf{L}p^*_1 E \otimes^{\mathbf{L}} K) && \text{([GW23, Remark 22.94 \& Theorem 22.99])} \\
&\cong \mathbf{R}q_{i2,*}(t^*_i\mathbf{L}p^*_1 E \otimes^{\mathbf{L}} t^*_i K) && \text{([Sta26, Tag 07A4])} \\
&\cong \mathbf{R}q_{i2,*}(\mathbf{L}(p_1 \circ t_i)^* E \otimes^{\mathbf{L}} t^*_i K) \\
&\cong \mathbf{R}q_{i2,*}(\mathbf{L}(s_{i1} \circ q_{i1})^* E \otimes^{\mathbf{L}} t^*_i K) && (p_1 \circ t_i = s_{i1} \circ q_{i1}) \\
&\cong \mathbf{R}q_{i2,*}(\mathbf{L}q^*_{i1}s^*_{i1} E \otimes^{\mathbf{L}} t^*_i K) \\
&\cong \Phi_{t^*_i K}(s^*_{i1}E).
\end{aligned}$$

Moreover, from [ELS20, Theorem 4.4], we have a Verdier localization $s^*_{i1}\colon D^b_{\mathrm{coh}}(Y_1) \to D^b_{\mathrm{coh}}(Y'_{i1})$. Hence, if coupled with our hypothesis, it follows that $\Phi_{t^*_i K}\colon D^b_{\mathrm{coh}}(Y'_{i1}) \to D_{\mathrm{qc}}(Y'_{i2})$ factors through $D^b_{\mathrm{coh}}(Y'_{i2})$ for each i.

Then, by Lemma 4.4, one has that $\mathbf{R}q_{i1,*}(t^*_i K \otimes^{\mathbf{L}} \operatorname{Perf}(Y'_{i1} \times_S Y'_{i2}))$ is contained in $\operatorname{Perf}(Y'_{i1})$ for each i. However, once more from the cube above, we have another computation based on similar reasoning for each E in $D_{\mathrm{qc}}(Y_1 \times_S Y_2)$:

$$\begin{aligned} s_{i1}^* \mathbf{R}p_{1,*}(K \otimes^{\mathbf{L}} E) &\cong \mathbf{R}q_{i1,*} t_i^*(K \otimes^{\mathbf{L}} E) \\ &\cong \mathbf{R}q_{i1,*}(t_i^* K \otimes^{\mathbf{L}} t_i^* E). \end{aligned}$$

There is, from [Nee92, Theorem 2.1], a Verdier localization sequence,

$$D_{\mathrm{qc},Y_1 \setminus Y'_{i1}}(Y_1) \to D_{\mathrm{qc}}(Y_1) \xrightarrow{s_{i1}^*} D_{\mathrm{qc}}(Y'_{i1}),$$

which induces a Verdier localization (up to summands) $s_{i1}^* \colon \operatorname{Perf}(Y_1) \to \operatorname{Perf}(Y'_{i1})$. It follows, for each P in $\operatorname{Perf}(Y_1 \times_S Y_2)$ and each i, that $s_{i1}^* \mathbf{R}p_{1,*}(K \otimes^{\mathbf{L}} P)$ is in $\operatorname{Perf}(Y'_{i1})$. This tells us any such object $\mathbf{R}p_{1,*}(K \otimes^{\mathbf{L}} P)$ must belong to $\operatorname{Perf}(Y_1)$ as desired. □

Theorem 4.6. *Let $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ be proper morphisms to a Noetherian scheme. Consider the fibered square:*

$$\begin{tikzcd} Y_1 \times_S Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & S. \end{tikzcd}$$

Then the following are equivalent for any object K in $D^-_{\mathrm{coh}}(Y_1 \times_S Y_2)$:

(1) Φ_K induces an exact functor $\breve{\Phi}_K \colon D_{\mathrm{sg}}(Y_1) \to D_{\mathrm{sg}}(Y_2)$
(2) $\mathbf{R}p_{i,*}(K \otimes^{\mathbf{L}} \operatorname{Perf}(Y_1 \times_S Y_2)) \subseteq \operatorname{Perf}(Y_i)$ for each i.

Proof. This follows from Propositions 4.2 and 4.5 □

5. Some consequences

Example 5.1. Let $s_i \colon Y_i \to \operatorname{Spec}(k)$ be proper varieties over a field k for $i = 1, 2$. Consider the fibered square:

$$\begin{tikzcd} Y_1 \times_S Y_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y_2 \arrow[d, "s_2"] \\ Y_1 \arrow[r, "s_1"'] & \operatorname{Spec}(k). \end{tikzcd}$$

Suppose A_1 is an object of $D^b_{\mathrm{coh}}(Y_1)$ and A_2 is an of object $D^b_{\mathrm{coh}}(Y_2)$ which is not in $\operatorname{Perf}(Y_2)$. Assume that $\mathbf{R}s_{1,*}(A_1 \otimes^{\mathbf{L}} D^b_{\mathrm{coh}}(Y_1))$ is contained in $D^b_{\mathrm{coh}}(\operatorname{Spec}(k))$ (e.g. A_1 is in $\operatorname{Perf}(Y_1)$).

Then a direct computation shows the following:

$$\begin{aligned}
\Phi_{p_1^*A_1\otimes^{\mathbf{L}} p_2^*A_2}(E) &:= \mathbf{R}p_{2,*}(p_1^*E \otimes^{\mathbf{L}} p_1^*A_1 \otimes^{\mathbf{L}} p_2^*A_2) \\
&\cong \mathbf{R}p_{2,*}(p_1^*E \otimes^{\mathbf{L}} p_1^*A_1) \otimes^{\mathbf{L}} A_2 && \text{([Sta26, Tag 08EU])} \\
&\cong \mathbf{R}p_{2,*}p_1^*(E \otimes^{\mathbf{L}} A_1) \otimes^{\mathbf{L}} A_2 && \text{([Sta26, Tag 07A4])} \\
&\cong s_2^*\mathbf{R}s_{1,*}(E \otimes^{\mathbf{L}} A_1) \otimes^{\mathbf{L}} A_2 && \text{([GW23, Thm. 22.99])} \\
&\cong \Big(\bigoplus_{n\in\mathbb{Z}} s_2^*\mathcal{O}_{\operatorname{Spec}(k)}^{\oplus r_n}[n]\Big) \otimes^{\mathbf{L}} A_2 \\
&\cong \Big(\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_{Y_2}^{\oplus r_n}[n]\Big) \otimes^{\mathbf{L}} A_2 \\
&\cong \bigoplus_{n\in\mathbb{Z}} A_2^{\oplus r_n}[n].
\end{aligned}$$

Observe, from our hypothesis on A_1, that $r_n \neq 0$ for at most finitely[1] many n. This tells us, as A_2 is not in $\operatorname{Perf}(Y_2)$,

$$\Phi_{p_1^*A_1\otimes^{\mathbf{L}} p_2^*A_2} \colon D^b_{\operatorname{coh}}(Y_1) \to D_{\operatorname{qc}}(Y_2)$$

factors through $D^b_{\operatorname{coh}}(Y_2)$. However,

$$\Phi_{p_1^*A_1\otimes^{\mathbf{L}} p_2^*A_2} \colon \operatorname{Perf}(Y_1) \to D_{\operatorname{qc}}(Y_2)$$

cannot factor through $\operatorname{Perf}(Y_2)$.

Example 5.2. Let X be a variety over a field.

(1) Consider the projectivization $p\colon \mathbb{P}_X(\mathcal{E}) \to X$ of finite locally free sheaf $\mathcal{E}$ on X. Then, by [BS20, Theorem 6.7] functors $\Phi_n \colon E \mapsto p^*E \otimes^{\mathbf{L}} \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(n)$ are integral transforms (with kernel $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(n)$) for each integer n. Moreover, [BS20, Corollary 6.8] ensures each Φ_n preserves both perfect complexes and those with bounded coherent cohomology.

(2) Suppose $i\colon Z \to X$ is a closed immersion that is regular (in the sense of [Sta26, Tag 0638]) of constant codimension $c \geq 0$. Denote by $f\colon \widetilde{X} \to X$ for the blowup of X along Z. Consider the following fibered square:

$$\begin{tikzcd}
E \arrow[r, "i'"] \arrow[d, "f'"'] & \widetilde{X} \arrow[d, "f"] \\
Z \arrow[r, "i"'] & X
\end{tikzcd}$$

where E is the exceptional divisor. Then, by [BS20, Theorem 6.9], the functors $\Phi_j \colon A \mapsto \mathcal{O}_{\widetilde{X}}(-j\cdot E) \otimes^{\mathbf{L}} \mathbf{R}i'_*\mathbf{L}(f')^*A$ are integral transforms if $j \leq 0$. Moreover, from [BS20, Corollary 6.10], if $-c+1 \leq j \leq 0$, one has that Φ_j preserves perfect complexes and objects with bounded coherent cohomology.

Remark 5.3. We remind ourselves of two facts.

---

[1] Any object of $D^b_{\operatorname{coh}}(\operatorname{Spec}(k))$ is isomorphic to an object of the form $\oplus_{t\in\mathbb{Z}}\mathcal{O}_{\operatorname{Spec}(k)}^{\oplus d_t}[t]$.

- Let $F\colon \mathcal{T} \to \mathcal{S}$ be an exact functor between triangulated categories. Assume $\mathcal{T}$ is compactly generated. Then F admits a right adjoint if, and only if, it preserves small coproducts. See [Nee96, Theorem 4.1].
- Let $F\colon \mathcal{S} \rightleftarrows \mathcal{T} \colon G$ be an adjoint pair of exact functors between triangulated categories. Assume $\mathcal{S}$ is compactly generated. Then F preserves compacts if, and only if, G preserves small coproducts. See [Nee96, Theorem 5.1].

Proposition 5.4. *With the notation of Theorem 4.6, assume additionally that S is affine. Then $\Phi_K\colon D_{\mathrm{qc}}(Y_1) \to D_{\mathrm{qc}}(Y_2)$ admits a left adjoint if, and only if, the functor $\Phi_K\colon D^b_{\mathrm{coh}}(Y_1) \to D_{\mathrm{qc}}(Y_2)$ factors through $D^b_{\mathrm{coh}}(Y_2)$. In such a situation, $\Phi_{K'}$ is the left adjoint to Φ_K on D_{qc}, where $K' := \mathbf{R}\mathcal{H}om(K, p_1^!\mathcal{O}_{Y_1})$.*

Proof. First, we prove the forward direction where Φ_K admits a left adjoint Φ as a functor on D_{qc}. Observe that Φ_K always admits a right adjoint as a functor on D_{qc} that is given by

$$E \mapsto \mathbf{R}p_{1,*}\mathbf{R}\mathcal{H}om(K, \mathbf{L}p_2^*E).$$

This means Φ_K preserves small coproducts, and so, Φ preserves compacts. In other words, $\Phi(\mathrm{Perf}(Y_2)) \subseteq \mathrm{Perf}(Y_1)$. Then Lemma 4.1 tells us $\Phi_K(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$ as desired, e.g. the restriction of Φ_K to $D^b_{\mathrm{coh}}(Y_1)$ agrees with the unique functor in Lemma 4.1 (indeed, [Nee21, Example 0.7] gives a triangulated equivalence between the category of finite cohomological functors on perfect complexes and the bounded derived category of bounded pseudocoherent complexes).

Next, we check the converse direction. It follows from Lemma 4.4 that $\mathbf{R}p_{1,*}(K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2))$ is contained in $\mathrm{Perf}(Y_1)$. There is a string of natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(E, \Phi_K(G)) &\cong \mathrm{Hom}(E, \mathbf{R}p_{2,*}(K \otimes^{\mathbf{L}} \mathbf{L}p_1^*G)) && \text{(Definition)}\\
&\cong \mathrm{Hom}(\mathbf{L}p_2^*E, K \otimes^{\mathbf{L}} \mathbf{L}p_1^*G) && \text{(Adjunction)}\\
&\cong \mathrm{Hom}(\mathbf{R}p_{1,*}(\mathbf{R}\mathcal{H}om(K, p_1^!\mathcal{O}_{Y_1}) \otimes^{\mathbf{L}} \mathbf{L}p_2^*E), G) && \text{([Bal09, Lemma 3.10] with } p_1)
\end{aligned}$$

This shows the desired claim. ☐

Remark 5.5. Recall that a variety X over a field k is called a *(birational) derived splinter* if the natural map $\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_Y$ splits for every proper (resp. birational) surjective morphism $f\colon Y \to X$. The notion of derived splinters was introduced by Bhatt [Bha12], while the birational variant originates in unpublished work of Kovács. Over characteristic zero the two notions coincide with having rational singularities (see [DLMV26]), but in positive characteristic birational derived splinters need not be derived splinters (see e.g. [DLMV26, Example 2.7]).

Proposition 5.6 (cf. [Ola23, Proposition 9]). *Let Y_1 and Y_2 be birational derived splinters that are proper over an uncountable field k. Consider the following situation:*

- *for each i there is a resolution of singularities $\pi_i\colon \widetilde{Y}_i \to Y_i$*
- *there is $K \in D^-_{\mathrm{coh}}(Y_1 \times_k Y_2)$ such that $\Phi_K\colon D^b_{\mathrm{coh}}(Y_1) \to D^b_{\mathrm{coh}}(Y_2)$ is fully faithful.*

If $\Phi_K(\mathrm{Perf}(Y_1)) \subseteq \mathrm{Perf}(Y_2)$, then $\dim Y_1 \leq \dim Y_2$.

Proof. First, one can augment the proof [DLM24, Proposition 6.7] for birational derived splinters as the same splitting condition holds on the unit morphisms. Now, if coupled with [Ola23, Lemma 7], we see that $\mathrm{Cdim}\, D^b_{\mathrm{coh}}(Y_i) = \dim Y_i$ for each i. Moreover, Φ_K always has a right adjoint Φ (see proof of Proposition 5.4). Then Lemma 4.1, coupled with our hypothesis, ensures Φ restricts to give an exact functor $D^b_{\operatorname{coh}}(Y_2) \to D^b_{\operatorname{coh}}(Y_1)$. However, Φ_K being fully faithful ensures $\Phi \circ \Phi_K \to \mathbf{1}$ is an isomorphism (see e.g. [Sta26, Tag 07RB]), which in turn implies that $D^b_{\operatorname{coh}}(Y_2) \to D^b_{\operatorname{coh}}(Y_1)$ is a Verdier localization. This immediately implies that $\operatorname{Cdim} D^b_{\operatorname{coh}}(Y_1) \leq \operatorname{Cdim} D^b_{\operatorname{coh}}(Y_2)$, and hence $\dim Y_1 \leq \dim Y_2$. □

Example 5.7. In characteristic zero, being a (birational) derived splinter is equivalent to having rational singularities; see [Kov00, Bha12] for details. Moreover, [Hir64b, Hir64a] ensures the existence of resolution of singularities in characteristic zero.

Proposition 5.8. *Let $f_1 \colon Y_1 \to S$ and $f_2 \colon Y_2 \to S$ be proper morphisms to a quasi-compact regular scheme where at least one such morphism is flat. Suppose Y_1 and Y_2 are Fourier–Mukai S-partners given by a kernel K in $D^b_{\operatorname{coh}}(Y_1 \times_S Y_2)$. If Y_1 or Y_2 is not regular, then $K \notin \operatorname{Perf}(Y_1 \times_S Y_2)$.*

Proof. We prove the claim by contradiction. That is, $K \in \operatorname{Perf}(Y_1 \times_S Y_2)$. The hypothesis is that Φ_K restricts to a triangulated equivalence $D^b_{\operatorname{coh}}(Y_1) \to D^b_{\operatorname{coh}}(Y_2)$. However, as one of the Y_i is not regular, [CNS25a, Corollary 5.9] tells us the other Y_j cannot be regular. Let G_i be a classical generator for $\operatorname{Perf}(Y_i)$. Denote by $\pi_i \colon Y_1 \times_S Y_2 \to Y_i$ the projection morphisms. Then [BV03, Lemma 3.4.1] tells us $\mathbf{L}\pi_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2$ is a classical generator for $\operatorname{Perf}(Y_1 \times_S Y_2)$. Choose an object E in $D^b_{\operatorname{coh}}(Y_2)$. There is E' in $D^b_{\operatorname{coh}}(Y_1)$ such that $\mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} K) \cong E$. Note that K being perfect means K is finitely built by $\mathbf{L}\pi_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2$. Observe that projection formula tells us

$$G_2 \otimes^{\mathbf{L}} E \cong \mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} K \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2).$$

Clearly, $\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} K \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2$ is finitely built by $\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} \mathbf{L}\pi_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2$. Hence, we have that

$$\begin{aligned} G_2 \otimes^{\mathbf{L}} E &\cong \mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} K \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2) \\ &\in \langle \mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} \mathbf{L}\pi_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2) \rangle. \end{aligned}$$

We have an isomorphism

$$\mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} \mathbf{L}\pi_1^* G_1 \otimes^{\mathbf{L}} \mathbf{L}\pi_2^* G_2) \cong \mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} \mathbf{L}\pi_1^* G_1) \otimes^{\mathbf{L}} G_2.$$

Since S is regular, $D^b_{\operatorname{coh}}(S) = \operatorname{Perf}(S)$. Let G be a classical generator for $\operatorname{Perf}(S)$. By flat base change,

$$\mathbf{R}\pi_{2,*}(\mathbf{L}\pi_1^* E' \otimes^{\mathbf{L}} \mathbf{L}\pi_1^* G_1) \in \langle \mathbf{L}f_2^* G \rangle.$$

However, $\mathbf{L}f_2^* G \in \operatorname{Perf}(Y_2) = \langle G_2 \rangle$. This tells us that $G_2 \otimes^{\mathbf{L}} E$ is finitely built by G_2; that is, $G_2 \otimes^{\mathbf{L}} E$ is perfect. But this is absurd as it implies $E \in \operatorname{Perf}(Y_2)$. To be precise, we have shown $\Phi_K(D^b_{\operatorname{coh}}(Y_1)) \subseteq \operatorname{Perf}(Y_2)$, and yet Φ_K restricts to give a triangulated equivalence $D^b_{\operatorname{coh}}(Y_1) \to D^b_{\operatorname{coh}}(Y_2)$. So being that Y_2 is not regular, we would obtain from the work above that $\operatorname{Perf}(Y_2) = D^b_{\operatorname{coh}}(Y_2)$, which is a contradiction. □

6. Preservation of adjointness

This subsection studies the behavior of adjoints for integral transforms under a change of base schemes. The following is part of our motivation. To avoid repeatedly stating similar constraints, the following is used as a placeholder throughout our work.

Setup 6.1. Let $t\colon T \to S$ be a morphism of Noetherian schemes. Suppose $f_i\colon Y_i \to S$ are proper flat morphisms with $i \in \{1,2\}$. Consider the commutative diagram

$$\begin{tikzcd}
Y_1 \times_S Y_2 \times_S T \arrow[rr, "g_1'"'] \arrow[dr, "g_2'"'] \arrow[dd, "t'"'] & & Y_2 \times_S T \arrow[dr, "g_2"] \arrow[dd, dashed, "t_2"' near end] & \\
& Y_1 \times_S T \arrow[rr, "g_1" near start] \arrow[dd, "t_1" near start] & & T \arrow[dd, "t"] \\
Y_1 \times_S Y_2 \arrow[rr, dashed, "f_1'" near start] \arrow[dr, "f_2'"'] & & Y_2 \arrow[dr, "f_2"] & \\
& Y_1 \arrow[rr, "f_1"'] & & S
\end{tikzcd} \tag{6.1}$$

obtained by base change along t.

Lemma 6.2. *Consider Setup 6.1. Then each face of Equation (6.1) is a tor-independent square (in the sense of* [Lip09, Definition 3.10.2]*).*

Proof. As each f_i, f_i', g_i, and g_i' are flat, the desired claim follows from [GW23, Remark 22.94]. □

Lemma 6.3. *Consider Setup 6.1 with t an affine morphism. If $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ is relatively perfect over Y_1 (resp. over Y_2), then $\Phi_{\mathbf{L}(t')^*K}$ restricts to an exact functor on D^b_{coh} (resp. on* Perf*).*

Proof. We only prove the case of K being relatively perfect over Y_2 because a similar argument applies for D^b_{coh}. As $\Phi_K(\mathrm{Perf}(Y_1)) \subseteq \mathrm{Perf}(Y_2)$, Proposition 4.2 implies

$$\mathbf{R}(f_1')_*(K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2)) \subseteq \mathrm{Perf}(Y_2).$$

By base change, we know that t_2, t_2, t' are affine morphisms. Choose $G \in \mathrm{Perf}(Y_1 \times_S Y_2)$ such that $\mathrm{Perf}(Y_1 \times_S Y_2) = \langle G \rangle$. Now, $\mathbf{L}(t')^*G$ satisfies

$$\mathrm{Perf}(Y_1 \times_S Y_2 \times_S T) = \langle \mathbf{L}(t')^*G \rangle,$$

see e.g. [Sta26, Tag 0BQT]. Also, via base change, we know that $g_1, g_2, f_2', f_1', g_2', g_1'$ are proper flat morphisms. Using flat base change (see e.g. [GW23, Remark 22.94 & Theorem 22.99]), we can tie things together and see that

$$\begin{aligned}
&\mathbf{R}(g_1')_*(\mathbf{L}(t')^*K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2 \times_S T)) \\
&\quad \subseteq \mathbf{R}(g_1')_*\langle \mathbf{L}(t')^*(K \otimes^{\mathbf{L}} G)\rangle \\
&\quad \subseteq \langle \mathbf{R}(g_1')_*\mathbf{L}(t')^*(K \otimes^{\mathbf{L}} G)\rangle \\
&\quad \subseteq \langle \mathbf{L}t_2^*\mathbf{R}(f_1')_*(K \otimes^{\mathbf{L}} G)\rangle \\
&\quad \subseteq \mathrm{Perf}(Y_2 \times_S T).
\end{aligned}$$

Therefore, by Proposition 4.2, $\Phi_{\mathbf{L}(t')^*K}$ restricts to an exact functor on Perf. □

Lemma 6.4. *Consider Setup 6.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Then on D_{qc} there is a natural isomorphism*

$$\beta^K\colon \mathbf{L}t_2^* \circ \Phi_K \to \Phi_{\mathbf{L}(t')^*K} \circ \mathbf{L}t_1^*.$$

Proof. This follows from the string of isomorphisms for each $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$:

$$\begin{aligned} \mathbf{L}t_2^*\Phi_K(E) &= \mathbf{L}t_2^*\mathbf{R}(f_1')_*(\mathbf{L}(f_2')^*E \otimes^{\mathbf{L}} K) \\ &\cong \mathbf{R}(g_1')_*\mathbf{L}(t')^*(\mathbf{L}(f_2')^*E \otimes^{\mathbf{L}} K) && \text{(flat base change)} \\ &\cong \mathbf{R}(g_1')_*(\mathbf{L}(t')^*\mathbf{L}(f_2')^*E \otimes^{\mathbf{L}} \mathbf{L}(t')^*K) && \text{(monoidality of } \mathbf{L}(-)^*\text{)} \\ &\cong \mathbf{R}(g_1')_*(\mathbf{L}(g_2')^*\mathbf{L}(t_1)^*E \otimes^{\mathbf{L}} \mathbf{L}(t')^*K) && \text{(pseudofunctoriality of } \mathbf{L}(-)^*\text{)} \\ &= \Phi_{\mathbf{L}(t')^*K}(\mathbf{L}t_1^*E). \end{aligned}$$

□

Lemma 6.5. *Let $f\colon Y \to X$ be a faithfully flat morphism of Noetherian schemes. An object $E \in D_{\mathrm{qc}}(X)$ belongs to $D^-_{\mathrm{coh}}(X)$ (resp.* $\operatorname{Perf}(X)$*) if, and only if, $\mathbf{L}f^*E \in D^-_{\mathrm{coh}}(Y)$ (resp. $\mathbf{L}f^*E \in$* $\operatorname{Perf}(Y)$*).*

Proof. This is a special case of [GW23, Proposition 22.52]. □

Notation 6.6. Given a scheme X and a point $p \in X$, denote by $\sigma_p\colon \operatorname{Spec}(\mathcal{O}_{X,p}) \to X$ and $i_p\colon \operatorname{Spec}(\kappa(p)) \to X$ for the natural morphisms. Note that the set theoretic image of σ_p is the intersection of all open subschemes U of X which contain p.

Lemma 6.7. *Let X be a scheme, $E \in D^b_{\mathrm{qc}}(X)$ be pseudocoherent, and $p \in X$. Then the following are equivalent:*

1. *There is an open immersion $j\colon U \to X$ such that $p \in U$ and $\mathbf{L}j^*E$ is perfect*
2. *$\mathbf{L}\sigma_p^*E$ is perfect*
3. *$\mathbf{L}i_p^*E$ is bounded.*

Proof. To see that (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3), use the natural factorizations of σ_p and of i_p. Specifically,

$$\operatorname{Spec}(\kappa(p)) \to \operatorname{Spec}(\mathcal{O}_{X,p}) \to U \xrightarrow{j} X.$$

Indeed, one uses the fact that derived pullback preserves perfect complexes (see e.g. [Sta26, Tag 09UA]).

Lastly, we check (3) $\Longrightarrow$ (1). In this case, we may assume X is affine. However, the result follows from [AJS23, Theorem 2.3, (iii) $\Longrightarrow$ (i)]. □

Proposition 6.8. *Let X be a quasi-separated quasi-compact scheme and $E \in D^b_{\mathrm{qc}}(X)$ be pseudocoherent. Then the following conditions are equivalent:*

1. *E is perfect*
2. *$\mathbf{L}i_p^*E$ is bounded for every $p \in X$*
3. *$\mathbf{L}i_p^*E$ is bounded for every $p \in X$ which is closed*
4. *$\mathbf{L}\sigma_p^*E$ is perfect for every $p \in X$*
5. *$\mathbf{L}\sigma_p^*E$ is perfect for every $p \in X$ which is closed.*

Proof. It is straightforward to see that (2) $\Longrightarrow$ (3). Also, by [AJS23, Theorem 2.3], we have (1) $\Longleftrightarrow$ (2). Moreover, from Lemma 6.7, we have (4) $\Longleftrightarrow$ (2) and (5) $\Longleftrightarrow$ (3). Let $x \in X$. Since X is quasi-compact, there is a closed point $x \in X$ such that $x \in \overline{\{p\}}$ [GW20, Exercise 3.13]. Hence, σ_p factors through the natural morphisn $\operatorname{Spec}(\mathcal{O}_{X,p}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$ with σ_x, and so (5) $\Longleftrightarrow$ (4) by [Sta26, Tag 09UA]. □

Lemma 6.9. *Consider Setup 6.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$.*

(1) *If* $\mathbf{L}(t')^*K$ *is relatively perfect over* $Y_1 \times_S T$ *(resp.* $Y_2 \times_S T$*) where* t *is affine and faithfully flat, then* K *is relatively perfect over* Y_1 *(resp.* Y_2*).*

(2) *If* $\mathbf{L}(t')^*K$ *is relatively perfect over* $Y_1 \times_S \mathrm{Spec}(k)$ *(resp.* $Y_2 \times_S \mathrm{Spec}(k)$*) for all morphisms* $\mathrm{Spec}(k) \to S$ *from a field, then* K *is relatively perfect over* Y_1 *(resp.* Y_2*).*

(3) *If* $\mathbf{L}(t')^*K$ *is relatively perfect over* $Y_1 \times_S \mathrm{Spec}(\kappa(p))$ *(resp.* $Y_2 \times_S \mathrm{Spec}(\kappa(p))$*) for all closed immersions* $\mathrm{Spec}(\kappa(p)) \to S$ *associated to a closed point* $p \in S$*, then* K *is relatively perfect over* Y_1 *(resp.* Y_2*).*

Proof. We only show the case for $\mathbf{L}(t')^*K$ being relatively perfect over $Y_1 \times_S T$. To show the other case, one can argue like below and use Lemma 6.5.

To start, we prove the first claim. Let $E \in D^b_{\mathrm{coh}}(Y_1)$. As t' is faithfully flat, it follows that $\mathbf{L}t_1^*E \in D^b_{\mathrm{coh}}(Y_1 \times_S T)$. By Lemma 6.4, we know that $\mathbf{L}t_2^*\Phi_K(E) \cong \Phi_{\mathbf{L}(t')^*K}(\mathbf{L}t_1^*E)$. From Lemma 6.5, the hypothesis implies $\Phi_K(E) \in D^-_{\mathrm{coh}}(Y_2)$. So, it suffices to show that $\mathscr{H}^j(\Phi_K(E)) = 0$ for all but finitely many $j \in \mathbb{Z}$. Our hypothesis ensures that $\Phi_{\mathbf{L}(t')K}(\mathbf{L}t_1^*E) \in D^b_{\mathrm{coh}}(Y_2 \times_S T)$. Then, as t_2 is flat, for each $j \in \mathbb{Z}$,

$$\begin{aligned}
\mathrm{supp}(\mathscr{H}^j(\Phi_K(E))) &= t_2(\mathrm{supp}(t_2^*\mathscr{H}^j(\Phi_K(E)))) && \text{([Sta26, Tag 056J])}\\
&= t_2(\mathrm{supp}(\mathbf{L}t_2^*\mathscr{H}^j(\Phi_K(E)))) && \text{(Flatness of } t_2\text{)}\\
&= t_2(\mathrm{supp}(\mathscr{H}^j(\mathbf{L}t_2^*\Phi_K(E)))) && \text{(Flatness of } t_2\text{)}.
\end{aligned}$$

Hence, the desired claim regarding bounded cohomology follows. Consequently, we have $\Phi_K(D^b_{\mathrm{coh}}(Y_1)) \subseteq D^b_{\mathrm{coh}}(Y_2)$, which implies K is relatively perfect over Y_1.

Next, we check the second claim. Choose any $p \in Y_1$. Denote by $t \colon \mathrm{Spec}(\kappa(f_1(p))) \to S$ for the natural morphism. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(p)) \arrow[dr, "h"] \arrow[drr, bend left, "h_2"] \arrow[ddr, bend right, "h_1"'] & & \\
& Y_1 \times_S \mathrm{Spec}(\kappa(f_1(p))) \arrow[r, "g_1"] \arrow[d, "t_1"'] & \mathrm{Spec}(\kappa(f_1(p))) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"] & S
\end{tikzcd}$$

where h_1 is the natural morphism. Using the hypothesis, it follows that

$$\begin{aligned}
&\mathbf{R}(g_2')_*(\mathbf{L}(t')^*(K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2)))\\
&\qquad \subseteq \mathbf{R}(g_2')_*(\mathbf{L}(t')^*K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2 \times_S \mathrm{Spec}(\kappa(f_1(p)))))\\
&\qquad \subseteq \mathrm{Perf}(Y_1 \times_S \mathrm{Spec}(\kappa(f_1(p)))).
\end{aligned}$$

There is a fibered square

$$\begin{tikzcd}
Y_1 \times_S Y_2 \times_S \mathrm{Spec}(\kappa(f_1(p))) \arrow[r, "g_2'"'] \arrow[d, "t'"'] & Y_1 \times_S \mathrm{Spec}(\kappa(f_1(p))) \arrow[d, "t_1"] \\
Y_1 \times_S Y_2 \arrow[r, "f_2'"'] & Y_1.
\end{tikzcd}$$

So, from flat base change, we have for all $P \in \operatorname{Perf}(Y_1 \times_S Y_2)$,

$$\begin{aligned}\mathbf{L}t_1^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P) &\cong \mathbf{R}(g_2')_*\mathbf{L}(t')^*(K \otimes^{\mathbf{L}} P)\\ &\in \operatorname{Perf}(Y_1 \times_S \operatorname{Spec}(\kappa(f_1(p)))).\end{aligned}$$

This implies that

$$\mathbf{L}h_1^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P) \cong \mathbf{L}h^*\mathbf{L}t_1^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P) \in D^b_{\operatorname{coh}}(\kappa(p)),$$

because the derived pullback of perfect complexes remain perfect (see e.g. [Sta26, Tag 09UA]). Using [AJS23, Theorem 2.3(iv)], we deduce that $\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P) \in \operatorname{Perf}(Y_1)$; note that $\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P)$ is bounded because $K \otimes^{\mathbf{L}} P$ is a bounded pseudocoherent complex and f_2' is proper. Indeed, we have shown that $\mathbf{L}b^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P)$ is bounded for all $q \in Y_1$ with natural morphism $b\colon \operatorname{Spec}(\kappa(q)) \to Y_1$. Thus, K is relatively perfect over Y_1.

Lastly, we check the third claim. However, we can argue essentially like above. In particular, the argument above shows that the stalks of $\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} P)$ at all closed points of Y_1 are perfect. Consequently, the desired claim follows from Proposition 6.8. □

Theorem 6.10. *Consider Setup 6.1. Let $K \in D^b_{\operatorname{coh}}(Y_1 \times_S Y_2)$. Then the following are equivalent:*

(1) K is relatively perfect over Y_1 (resp. Y_2)

(2) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$ (resp. $Y_2 \times_S T$) for every affine morphism t from a Noetherian scheme

(3) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$ (resp. $Y_2 \times_S \operatorname{Spec}(k)$) for all morphisms $t\colon \operatorname{Spec}(k) \to S$ from a field

(4) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(\kappa(p))$ (resp. $Y_2 \times_S \operatorname{Spec}(\kappa(p))$) for all closed immersions $t\colon \operatorname{Spec}(k(p)) \to S$ associated to a closed point $p \in S$

(5) $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$ (resp. $Y_2 \times_S T$) for some affine surjective morphism $t\colon T \to S$.

Proof. We only prove the case of being relatively perfect over Y_1 because the other can be shown analogously. By Lemma 6.3, we see that (1) implies (2). Moreover, we know that (2) implies (3) by [Sta26, Tag 01SI]. Clearly, (3) implies (4). Furthermore, to see (4) implies (5), use that Lemma 6.9(3) tells us K is relatively perfect over Y_1, i.e. take $t = 1_S$.

So, we need to check (5) implies (1). Let $t\colon T \to S$ be an affine surjective morphism such that $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$. Choose any morphism $s\colon \operatorname{Spec}(k) \to S$ from a field. Consider the fiber product

$$\begin{tikzcd}
T \times_S \operatorname{Spec}(k) \arrow[r, "t'"] \arrow[d, "s'"'] & \operatorname{Spec}(k) \arrow[d, "s"] \\
T \arrow[r, "t"'] & S.
\end{tikzcd}$$

By base change, s' is affine and t' is an affine surjection. To avoid complicated diagrams involving base changes, we need some notation. Specifically, let $K_\#$ be the derived pullback of K along the natural morphism $Y_1 \times_S Y_2 \times_S \# \to Y_1 \times_S Y_2$ for $\#$ any scheme appearing in the diagram above. Now, the hypothesis tells us that K_T is relatively perfect over $Y_1 \times_S T$. Hence, Lemma 6.3 implies $K_{T \times_S \operatorname{Spec}(k)}$ is relatively perfect over $Y_1 \times_S T \times_S \operatorname{Spec}(k)$ because s' is affine. However, t' is affine and faithfully flat, so Lemma 6.9(1) ensures that $K_{\operatorname{Spec}(k)}$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(k)$. However, s was an arbitrary morphism from a field, so Lemma 6.9(2) tells us K is relatively perfect over Y_1 as desired. □

Lemma 6.11 (Base change for relative dualizing complex). *Let S be a quasi-compact quasi-separated scheme. Consider a fibered square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & S' \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

where f is proper, flat, and of finite presentation. Then $\mathbf{L}(g')^* f^! \mathcal{O}_S$ is a relative dualizing complex for f'. In fact, there is an isomorphism $\mathbf{L}(g')^* f^! \mathcal{O}_S \to (f')^! \mathcal{O}_{S'}$.

Proof. This is known but we include it for convenience. By [Sta26, Tag 0B6S & Tag 0E2Z], we know that $(f')^! \mathcal{O}_{S'}$ and $f^! \mathcal{O}_S$ are respectively relative dualizing complexes for f' and f. Moreover, [Sta26, Tag 0E2Y] tells us that $\mathbf{L}(g')^* f^! \mathcal{O}_S$ is a relative dualizing complex for f'. However, [Sta26, Tag 0E2W] implies relative dualizing complexes are unique up to isomorphism, and so $\mathbf{L}(g')^* f^! \mathcal{O}_S \cong (f')^! \mathcal{O}_{S'}$. ☐

Proposition 6.12. *Consider Setup 6.1 with t affine. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over Y_2. Then $\Phi_{\mathbf{L}(t')^* K'}$ is right adjoint to $\Phi_{\mathbf{L}(t')^* K}$ on D_{qc} where $K' := \mathbf{R}\mathcal{H}om(K, (f_1')^! \mathcal{O}_{Y_2})$. In both cases, $K' \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$.*

Proof. That the last claim holds follows from [Riz17, Definition 2.11 & Lemma 2.12]. So, we check the first claim. For any $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$ and $A \in D_{\mathrm{qc}}(Y_2 \times_S T)$, there is a string of natural isomorphisms obtained by adjunctions:

$$\begin{aligned}
&\mathrm{Hom}(\Phi_{\mathbf{L}(t')^* K}(E), A) \\
&\quad = \mathrm{Hom}(\mathbf{R}(g_1')_*(\mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} \mathbf{L}(t')^* K), A) && \text{(Definition)} \\
&\quad \cong \mathrm{Hom}(\mathbf{L}(g_2')^* E \otimes^{\mathbf{L}} \mathbf{L}(t')^* K, (g_1')^! A) && \text{(see e.g. [Sta26, Tag 0A9E])} \\
&\quad \cong \mathrm{Hom}(\mathbf{L}(g_2')^* E, \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! A)) && \text{(see e.g. [Sta26, Tag 08DH])} \\
&\quad \cong \mathrm{Hom}(E, \mathbf{R}(g_2')_* \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! A)).
\end{aligned}$$

The desired claim follows if we can find an isomorphism

$$\begin{aligned}
&\mathbf{R}(g_2')_* \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! A) \\
&\quad \cong \mathbf{R}(g_2')_* \Big(\mathbf{L}(t')^* (\mathbf{R}\mathcal{H}om(K, (f_1')^! \mathcal{O}_{Y_2})) \otimes^{\mathbf{L}} \mathbf{L}(g_1')^* A \Big) =: \Phi_{\mathbf{L}(t')^* K'}(A).
\end{aligned}$$

Moreover, $(f_1')^! \mathcal{O}_{Y_2} \in D^+_{\mathrm{coh}}(Y_1 \times_S Y_2)$ as upper shriek functors for proper morphisms preserve complexes with bounded below and coherent cohomology (see e.g. [Sta26, Tag 0AU1]). So, we have a string of isomorphisms

$$\begin{aligned}
&\mathbf{L}(t')^* \mathbf{R}\mathcal{H}om(K, (f_1')^! \mathcal{O}_{Y_2}) \\
&\quad \cong \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, \mathbf{L}(t')^* (f_1')^! \mathcal{O}_{Y_2}) && \text{(Lemma 2.8)} \\
&\quad \cong \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^* K, (g_1')^! \mathcal{O}_{Y_2 \times_S T}) && \text{(Lemma 6.11)}.
\end{aligned}$$

By Lemma 6.3, we see that $\Phi_{\mathbf{L}(t')^* K}$ restricts to an exact functor $\mathrm{Perf}(Y_1 \times_S T) \to \mathrm{Perf}(Y_2 \times_S T)$. Then Proposition 4.2 implies

$$\mathbf{R}(g_1')_*(\mathbf{L}(t')^* K \otimes^{\mathbf{L}} \mathrm{Perf}(Y_1 \times_S Y_2 \times_S T)) \subseteq \mathrm{Perf}(Y_2 \times_S T).$$

There is another string of isomorphisms for all $A \in D_{\mathrm{qc}}$:

$$\begin{aligned}
&\mathbf{R}(g_2')_*\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K,(g_1')^!A)\\
&\quad\cong \mathbf{R}(g_2')_*\mathbf{R}\mathcal{H}om\Big(\mathbf{L}(t')^*K,\mathbf{L}(g_1')^*A\otimes^{\mathbf{L}}(g_1')^!\mathcal{O}_{Y_2\times_S T}\Big) \qquad \text{([Sta26, Tag 0B6S])}\\
&\quad\cong \mathbf{R}(g_2')_*\Big(\mathbf{R}\mathcal{H}om\left(\mathbf{L}(t')^*K,(g_1')^!\mathcal{O}_{Y_2\times_S T}\right)\otimes^{\mathbf{L}}\mathbf{L}(g_1')^*A\Big)\\
&\quad\cong \mathbf{R}(g_2')_*\Big(\mathbf{L}(t')^*\mathbf{R}\mathcal{H}om(K,(f_1')^!\mathcal{O}_{Y_2})\otimes^{\mathbf{L}}\mathbf{L}(g_1')^*A\Big).
\end{aligned}$$

The last isomorphism comes from the work above, whereas the second isomorphism is [Riz17, Lemma 2.13] together with the fact that $\mathbf{L}(t')^*K$ is g_1'-perfect via Theorem 6.10, which completes the proof. ☐

Corollary 6.13. *Consider Setup 6.1. Let $K \in D^b_{\mathrm{coh}}(Y_1\times_S Y_2)$ be relatively perfect over each Y_i. Denote by K' for the kernel of the integral transform obtained in Proposition 6.12 which is right adjoint to Φ_K on D_{qc}. Then $\Phi_{K'}$ restricts to D^b_{coh}.*

Proof. Since K is p_2-perfect, [Bal09, Lemma 3.7] implies that K' is p_2-perfect. Hence, by Proposition 4.5, $\Phi_{K'}(D^b_{\mathrm{coh}}(\mathcal{Y}_2))\subseteq D^b_{\mathrm{coh}}(\mathcal{Y}_1)$. ☐

Lemma 6.14. *Let $f_i\colon Y_i\to S$ be proper flat morphisms to a Noetherian scheme where $i\in\{1,2\}$. Denote by $f_i'\colon Y_1\times_S Y_2\to Y_i$ the natural morphisms. Suppose $K\in D^b_{\mathrm{coh}}(Y_1\times_S Y_2)$ is relatively perfect over Y_1. Then Φ_K admits a left adjoint on D_{qc}. In particular, the left adjoint is of the form $\Phi_{K'}$ where $K' := \mathbf{R}\mathcal{H}om(K,(f_2')^!\mathcal{O}_{Y_1})$.*

Proof. The desired claim follows from the string of natural isomorphisms for all $E\in D_{\mathrm{qc}}(Y_2)$ and $G\in D_{\mathrm{qc}}(Y_1)$:

$$\begin{aligned}
\operatorname{Hom}(E,\Phi_K(G)) &\cong \operatorname{Hom}(E,\mathbf{R}(f_1')_*(K\otimes^{\mathbf{L}}\mathbf{L}(f_2')^*G)) && \text{(Definition)}\\
&\cong \operatorname{Hom}(\mathbf{L}(f_1')^*E,K\otimes^{\mathbf{L}}\mathbf{L}(f_2')^*G) && \text{(Adjunction)}\\
&\cong \operatorname{Hom}(\mathbf{R}(f_2')_*(\mathbf{R}\mathcal{H}om(K,(f_2')^!\mathcal{O}_{Y_1})\otimes^{\mathbf{L}}\mathbf{L}(f_1')^*E),G) && \text{([Bal09, Lemma 3.10])}.
\end{aligned}$$

☐

Proposition 6.15. *Consider Setup 6.1 with t affine. Let $K\in D^b_{\mathrm{coh}}(Y_1\times_S Y_2)$ be relatively perfect over Y_1. Then $\Phi_{\mathbf{L}(t')^*K'}$ is left adjoint to $\Phi_{\mathbf{L}(t')^*K}$ on D_{qc} where $K' := \mathbf{R}\mathcal{H}om(K,(f_2')^!\mathcal{O}_{Y_1})$. In such a case, $K'\in D^b_{\mathrm{coh}}(Y_1\times_S Y_2)$.*

Proof. That the last claim holds follows from [Riz17, Definition 2.11 & Lemma 2.12]. So, we check the first claim. By Lemma 6.3, $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1\times_S T$. Moreover, from Lemma 6.14, the kernel of the integral transform which is left adjoint to $\Phi_{\mathbf{L}(t')^*K}$ is given by the object $\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K,(g_2')^!\mathcal{O}_{Y_1\times_S T})$. So, it suffices to show there is an isomorphism

$$\begin{aligned}
&\mathbf{R}(g_2')_*(\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K,(g_2')^!\mathcal{O}_{Y_1\times_S T})\otimes^{\mathbf{L}}\mathbf{L}(g_1')^*E)\\
&\qquad\to \mathbf{R}(g_2')_*(\mathbf{L}(t')^*\mathbf{R}\mathcal{H}om(K,(f_2')^!\mathcal{O}_{Y_1})\otimes^{\mathbf{L}}\mathbf{L}(g_1')^*E).
\end{aligned}$$

This follows if we can find an isomorphism

$$\mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K,(g_2')^!\mathcal{O}_{Y_1\times_S T})\to \mathbf{L}(t')^*(\mathbf{R}\mathcal{H}om(K,(f_2')^!\mathcal{O}_{Y_1})).$$

As in Lemma 6.2, $(g_2')^!\mathcal{O}_{Y_1\times_S T} \in D^+_{\mathrm{coh}}(Y_1 \times_S Y_2 \times_S T)$ because upper shriek functors for proper morphisms preserve complexes with bounded below and coherent cohomology (see e.g. [Sta26, Tag 0AU1]). Hence, we have a string of isomorphisms

$$\begin{aligned}
&\mathbf{L}(t')^*\mathbf{R}\mathcal{H}om(K,(f_2')^!\mathcal{O}_{Y_1})\\
&\qquad\cong \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K,\mathbf{L}(t')^*(f_2')^!\mathcal{O}_{Y_1}) \quad (\text{Lemma 2.8})\\
&\qquad\cong \mathbf{R}\mathcal{H}om(\mathbf{L}(t')^*K,(g_2')^!\mathcal{O}_{Y_1\times_S T}) \qquad (\text{Lemma 6.11}).
\end{aligned}$$

This completes the proof. □

Theorem 6.16. *Let S be a Noetherian scheme. Suppose Y_1 and Y_2 are proper and flat S-schemes. Consider $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Then the following are equivalent:*

1. *K is relatively perfect over Y_1 (resp. Y_2)*
2. *$\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S T$ (resp. $Y_2 \times_S T$) for any affine morphism t from a Noetherian scheme*
3. *$\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \mathrm{Spec}(k)$ (resp. $Y_2 \times_S \mathrm{Spec}(k)$) for all morphisms $t\colon \mathrm{Spec}(k) \to S$ from a field*
4. *$\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \mathrm{Spec}(\kappa(p))$ (resp. $Y_2 \times_S \mathrm{Spec}(\kappa(p))$) for every closed point $p \in S$ with associated closed immersion $t\colon \mathrm{Spec}(\kappa(p)) \to S$*
5. *$\mathbf{L}(t')^*K$ is relatively perfect over $Y_1\times_S T$ (resp. $Y_2\times_S T$) for some affine surjective morphism $t\colon T \to S$.*

Here, $t'\colon Y_1\times_S Y_2\times_S T \to Y_1\times_S Y_2$ is the natural morphism. In fact, in either case the corresponding left or right adjoints are preserved.

Proof. This follows from Theorem 6.10 together with Propositions 6.12 and 6.15. □

7. Fully faithfulness & equivalences

This subsection is concerned with the behavior of fully faithfulness or an equivalence for an integral transform under a change of base scheme.

Lemma 7.1 ([Mac78, IV.7, Exercise 4]). *Consider a diagram of functors and adjoint functors*

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "H"] \arrow[d, shift right, "F"'] & \mathscr{C}' \arrow[d, shift right, "F'"'] \\
\mathscr{D} \arrow[r, "K"'] \arrow[u, shift right, "G"'] \arrow[u, phantom, "\dashv"] & \mathscr{D}'. \arrow[u, shift right, "G'"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd} \tag{7.1}$$

Let $\beta\colon F'H \Rightarrow KF$ and $\alpha\colon HG \Rightarrow G'K$ be a pair of natural transformations. Then the following conditions are equivalent:

1. *The following diagram commutes:*

$$\begin{tikzcd}
F'HG \arrow[r, "\beta G"] \arrow[d, "F'\alpha"'] & KFG \arrow[d, "K\varepsilon"] \\
F'G'K \arrow[r, "\varepsilon' K"'] & K.
\end{tikzcd}$$

(2) The following diagram commutes:

$$\begin{tikzcd}
H \arrow[r, "\eta' H"] \arrow[d, "H\eta"'] & G'F'H \arrow[d, "G'\beta"] \\
HGF \arrow[r, "\alpha F"'] & G'KF.
\end{tikzcd}$$

(3) For all objects $c \in \mathscr{C}$, $d \in \mathscr{D}$, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{D}(Fc,d) \arrow[r, "\cong"] \arrow[d, "K"'] & \mathscr{C}(c,Gd) \arrow[d, "H"] \\
\mathscr{D}'(KFc,Kd) \arrow[d, "\alpha_c^*"'] & \mathscr{C}'(Hc,HGd) \arrow[d, "\beta_{d,*}"] \\
\mathscr{D}'(F'Hc,Kd) \arrow[r, "\cong"] & \mathscr{C}'(Hc,G'Kd).
\end{tikzcd}$$

Proof. First, we show (7.1(2)) $\implies$ (7.1(3)). Let $f\colon Fc \to d$ be a morphism. In the diagram of (7.1(3)), f is assigned along the top right composite as

$$f \mapsto Gf \cdot \eta_c \mapsto H(Gf \cdot \eta_c) \mapsto \alpha_d \cdot H(Gf \cdot \eta_c) = \alpha_d \cdot HGf \cdot H\eta_c.$$

Moreover, f is assigned along the left bottom composite as

$$f \mapsto Kf \mapsto Kf \cdot \beta_c \mapsto G'(Kf \cdot \beta_c) \cdot \eta'_{Hc} = G'Kf \cdot G'\beta_c \cdot \eta'_{Hc}.$$

So, by the naturality of α and (7.1(2)), we have

$$G'Kf \cdot G'\beta_c \cdot \eta'_{Hc} = G'Kf \cdot \alpha_{Fc} \cdot H\eta_c = \alpha_d \cdot HGf \cdot H\eta_c.$$

Next, we check that (7.1(3)) $\implies$ (7.1(2)). For $d = Fc$ and $f = 1_{Fc}$, we have equality between the last morphisms in the composites above. That is, it holds that $\alpha_{Fc} \cdot H\eta_c = G'\beta_c \cdot \eta'_{Hc}$, which gives the desired implication.

To check that (7.1(1)) $\iff$ (7.1(3)), one may argue like above because it is essentially dual. □

Definition 7.2. A **(lax) morphism of adjunctions** from $F \dashv G$ to $F' \dashv G'$ is comprised of a pair of functors

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "H"] \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] & \mathscr{C}' \arrow[d, shift right, "F'"'] \arrow[d, phantom, "\dashv"] \\
\mathscr{D} \arrow[r, "K"'] \arrow[u, shift right, "G"'] & \mathscr{D}' \arrow[u, shift right, "G'"']
\end{tikzcd}$$

plus a pair of natural transformations $\beta\colon F'H \Rightarrow KF$, $\alpha\colon HG \Rightarrow G'K$ such that the equivalent conditions in Lemma 7.1 hold. We call β (resp. α) the **left** (resp. **right**) **comparison transformation**. Furthermore, a **strict morphism of adjunctions** is just a lax morphism of adjunctions such that $F'H = KF$, $HG = G'K$, and β, α are the identity transformation.

Remark 7.3. In the category theory literature, a 'morphism of adjunctions' is usually referred in its strict sense [Mac78, IV.7] and [Rie17, Exercise 4.2.v], but a 'lax' one is instead described by saying that β and α are *mates* under Equation (7.1). See [CGR14, Sect. 1] for the precise definition of mates. Indeed, the mate $\overline{\beta}$ of β is by definition the natural transformation

$$\overline{\beta}\colon HG \xrightarrow{\eta'_{HG}} G'F'HG \xrightarrow{G'\beta_G} G'KFG \xrightarrow{G'K\varepsilon} G'K.$$

Therefore, $\overline{\beta}$ will equal α if, and only if, the corresponding adjunct morphisms $F'HG \to K$ (with respect to $F' \dashv G'$) are equal. This is exactly Lemma 7.1(1). Dually, one also sees that Lemma 7.1(2) amounts to the mate $\overline{\alpha}$ of α being equal to β.

Lemma 7.4. *Consider Setup 6.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Then on D_{qc} there is a natural isomorphism*

$$\alpha^K\colon \Phi_K \circ \mathbf{R}(t_1)_* \to \mathbf{R}(t_2)_* \circ \Phi_{\mathbf{L}(t')^*K}.$$

Proof. This follows from the string of natural isomorphisms for each $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$:

$$\begin{aligned}
\Phi_K \circ \mathbf{R}(t_1)_*(E) &= \mathbf{R}(f_1')_*\big(\mathbf{L}(f_2')^*\mathbf{R}(t_1)_*E \otimes^{\mathbf{L}} K\big) \\
&\cong \mathbf{R}(f_1')_*\big(\mathbf{R}t'_*\mathbf{L}(g_2')^*E \otimes^{\mathbf{L}} K\big) && \text{(flat base change)} \\
&\cong \mathbf{R}(f_1')_*\mathbf{R}t'_*\big(\mathbf{L}(g_2')^*E \otimes^{\mathbf{L}} \mathbf{L}(t')^*K\big) && \text{(projection formula)} \\
&\cong \mathbf{R}(t_2)_*\mathbf{R}(g_1')_*\big(\mathbf{L}(g_2')^*E \otimes^{\mathbf{L}} \mathbf{L}(t')^*K\big) && \text{(pseudofunctoriality of } \mathbf{R}(-)_*\text{)} \\
&= \mathbf{R}(t_2)_* \circ \Phi_{\mathbf{L}(t')^*K}(E).
\end{aligned}$$

□

Lemma 7.5. *Consider Setup 6.1. Let $K \in D_{\mathrm{qc}}(Y_1 \times_S Y_2)$. Then we have a morphism of adjoint pairs as depicted in the diagram*

$$\begin{tikzcd}
D_{\mathrm{qc}}(Y_1) \arrow[r, "\Phi_K"] \arrow[d, shift right=1.5ex, "\mathbf{L}t_1^*"'] \arrow[d, phantom, "\dashv"] & D_{\mathrm{qc}}(Y_2) \arrow[d, shift right=1.5ex, "\mathbf{L}t_2^*"'] \arrow[d, phantom, "\dashv"] \\
D_{\mathrm{qc}}(Y_1 \times_S T) \arrow[r, "\Phi_{\mathbf{L}(t')^*K}"'] \arrow[u, shift right=1.5ex, "\mathbf{R}(t_1)_*"'] & D_{\mathrm{qc}}(Y_2 \times_S T) \arrow[u, shift right=1.5ex, "\mathbf{R}(t_2)_*"']
\end{tikzcd}$$

with left and right comparison transformations given respectively by the isomorphisms β^K, α^K of Lemmas 6.4 and 7.4.

Proof. Denote by $\xi^i\colon \mathbf{L}t_i^*\mathbf{R}(t_i)_* \to \mathrm{id}$ for the counit of the adjunction $\mathbf{L}t_i^* \dashv \mathbf{R}(t_i)_*$ with $i = 1, 2$. To prove the desired claim, it suffices to verify that Lemma 7.1(1) holds; that is, that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{L}t_2^*\Phi_K\mathbf{R}(t_1)_* \arrow[r, "\beta^K_{\mathbf{R}(t_1)_*}"] \arrow[d, "\mathbf{L}t_2^*(\alpha^K)"'] & \Phi_{\mathbf{L}(t')^*K}\mathbf{L}t_1^*\mathbf{R}(t_1)_* \arrow[d, "\Phi_{\mathbf{L}(t')^*K}(\xi^1)"] \\
\mathbf{L}t_2^*\mathbf{R}(t_2)_*\Phi_{\mathbf{L}(t')^*K} \arrow[r, "\xi^2_{\Phi_{\mathbf{L}(t')^*K}}"'] & \Phi_{\mathbf{L}(t')^*K}.
\end{tikzcd}$$

For brevity sake, in the remainder of the proof, we drop the **L**'s and **R**'s in the notation for the derived functors (i.e. all functors now are understood to be derived). Hence, the diagram above is now

$$\begin{tikzcd}
 & t_2^*(t_2)_*\Phi_{(t')^*K} \arrow[dr, "\xi^2_{\Phi_{(t')^*K}}"] & \\
t_2^*\Phi_K(t_1)_* \arrow[ur, "t_2^*(\alpha^K)"] \arrow[dr, swap, "\beta^K_{(t_1)_*}"] & & \Phi_{(t')^*K}. \\
 & \Phi_{(t')^*K}t_1^*(t_1)^* \arrow[ur, swap, "\Phi_{(t')^*K}(\xi^1)"] &
\end{tikzcd}\tag{7.2}$$

Choose $E \in D_{\mathrm{qc}}(Y_1 \times_S T)$. If we expand the definition of α^K and β^K in (7.2) (see the proofs of Lemmas 6.4 and 7.4), we obtain a large diagram consisting of various faces. In what follows, we make this a bit more explicit. Particularly, we spell out each face needed to understand (7.2). To describe said diagram more explicitly, we use the following abbreviations for the (natural) isomorphisms:

- BC_i, $i \in \{1,2\}$ for the flat base change isomorphisms
- PF for the pojection formula
- $F_{(-)_*}$ for functoriality of $(-)_*$
- $F_{(-)^*}$ for functoriality of $(-)^*$
- $M_{(-)^*}$ for monoidality of $(-)^*$
- ξ' for the counit of $(t')^* \dashv (t')_*$.

Now, consider the following diagrams:

$$\begin{tikzcd}
t_2^*(f_1')_*((f_2')_*(t_1)_*E \otimes K) \arrow[r, "\mathrm{BC}_1"] \arrow[d, "\mathrm{BC}_2"] & t_2^*(f_1')_*((t')_*(g_2')^*E \otimes K) \arrow[d, "\mathrm{BC}_2"] \\
(g_1')_*(t')^*((f_2')^*(t_1)_*E \otimes K) \arrow[r, "\mathrm{BC}_1"] & (g_1')_*(t')^*(t'_*(g_2')^*E \otimes K)
\end{tikzcd}\tag{7.3}$$

$$\begin{tikzcd}
t_2^*(f_1')_*((t')_*(g_2')^*E \otimes K) \arrow[r, "\mathrm{PF}"] \arrow[d, "\mathrm{BC}_2"] & t_2^*(f_1')_*(t')_*((g_2')^*E \otimes (t')^*K) \arrow[d, "\mathrm{BC}_2"] \\
(g_1')_*(t')^*(t'_*(g_2')^*E \otimes K) \arrow[r, "\mathrm{PF}"] & (g_1')_*(t')^*t'_*((g_2')^*E \otimes (t')^*K)
\end{tikzcd}\tag{7.4}$$

$$\begin{tikzcd}
(g_1')_*(t')^*((f_2')^*(t_1)_*E \otimes K) \arrow[r, "\mathrm{BC}_1"] \arrow[d, "M_{(-)^*}"] & (g_1')_*(t')^*(t'_*(g_2')^*E \otimes K) \arrow[d, "M^*_{(-)}"] \\
(g_1')_*((t')^*(f_2')^*(t_1)_*E \otimes (t')^*K) \arrow[r, "\mathrm{BC}_1"] & (g_1')_*((t')^*t'_*(g_2')^*E \otimes (t')^*K)
\end{tikzcd}\tag{7.5}$$

(7.6)

$$\begin{tikzcd}
(g_1')_*((t')^*t'_*(g_2')^*E \otimes (t')^*K) \arrow[d, "\xi'"] & (g_1')_*(t')^*(t'_*(g_2')^*E \otimes K) \arrow[l, "M^*_{(-)}"'] \arrow[d, "\mathrm{PF}"] \\
(g_1')_*((g_2')^*E \otimes (t')^*K) & (g_1')_*(t')^*t'_*((g_2')^*E \otimes (t')^*K) \arrow[l, "\xi'"']
\end{tikzcd}$$

(7.7)

$$\begin{tikzcd}
t_2^*(f_1')_*(t')_*((g_2')^*E \otimes (t')^*K) \arrow[r, "F_{(-)^*}"] \arrow[d, "\mathrm{BC}_2"] & t_2^*(t_2)_*(g_1')_*((g_2')^*E \otimes (t')^*K) \arrow[d, "\xi^2"] \\
(g_1')_*(t')^*t'_*((g_2')^*E \otimes (t')^*K) \arrow[r, "\xi'"] & (g_1')_*((g_2')^*E \otimes (t')^*K)
\end{tikzcd}$$

(7.8)

$$\begin{tikzcd}
(g_1')_*((t')^*(f_2')^*(t_1)_*E \otimes (t')^*K) \arrow[r, "\mathrm{BC}_1"] \arrow[d, "F_{(-)^*}"] & (g_1')_*((t')^*t'_*(g_2')^*E \otimes (t')^*K) \arrow[d, "\xi'"] \\
(g_1')_*((g_2')^*t_1^*(t_1)_*E \otimes (t')^*K) \arrow[r, "\xi^1"] & (g_1')_*((g_2')^*E \otimes (t')^*K)
\end{tikzcd}$$

Assume that we have shown (7.3), (7.4), (7.5), (7.6), (7.7), (7.8) are commutative. Then (7.2) (in the case of E) is the pasting of these diagrams, i.e. gives the desired morphism

$$t_2^*(f_1')_*((f_2')_*(t_1)_*E \otimes K) \longrightarrow (g_1')_*((g_2')^*E \otimes (t')^*K).$$

So, let us explain why (7.3), (7.4), (7.5), (7.6), (7.7), (7.8) are commutative. As for (7.3), one may use naturality of BC_2, whereas (7.4) is due to the naturality of BC_2. Also, for (7.5), one can use the naturality of $M_{(-)^*}$.

Next, we explain (7.6). Observe that it is the functor $(g_1')_*$ applied to the following diagram evaluated at $((g_2')^*E, K)$:

$$\begin{tikzcd}
(t')^*(t'_*(-) \otimes -) \arrow[r, "\mathrm{PF}"] \arrow[d, "M_{(-)^*}"'] & (t')^*t'_*(- \otimes (t')^*(-)) \arrow[d, "\xi'"] \\
(t')^*t'_*(-) \otimes (t')^*(-) \arrow[r, "\xi'"'] & (-) \otimes (t')^*(-)
\end{tikzcd}$$

which commutes by definition of the projection formula, see e.g. proof of [GW23, Proposition 22.81].

Now, for (7.7). Note that it is the following diagram evaluated at $(g_2')^*E \otimes (t')^*K$:

$$\begin{tikzcd}
t_2^*(f_1')_*t'_* \arrow[r, "F_{(-)_*}"] \arrow[d, "\mathrm{BC}_2"'] & t_2^*(t_2)_*(g_1')_* \arrow[d, "\xi_2"] \\
(g_1')_*(t')^*t'_* \arrow[r, "\xi'"'] & (g_1')_*
\end{tikzcd}$$

which commutes by definition of the base change morphism. See [GW23, Definition and Remark 21.129] or [Lip09, (3.7.2)].

Lastly, we check (7.8). However, it is the functor $(g_1')_*(- \otimes (t')^*K)$ applied to the following diagram evaluated at E:

$$\begin{tikzcd}
(t')^*t'_*(g_2')^* \arrow[r, "\xi'"] & (g_2')^* \\
(t')^*(f_2')^*(t_1)_* \arrow[u, "\mathrm{BC}_1"] \arrow[r, "F_{(-)^*}"'] & (g_2')^*t_1^*(t_1)_* \arrow[u, "\xi_1"']
\end{tikzcd}$$

which commutes by definition of the base change morphism. □

Remark 7.6. Assume the hypothesis of Lemma 7.5. Denote by ζ^i for the unit of $t_i^* \dashv (t_i)_*$ if $i \in \{1,2\}$. From Lemmas 7.1 and 7.5, we have that the following diagram is commutative:

$$\begin{tikzcd}
\Phi_K \arrow[r, "\zeta^2_{\Phi_K}"] \arrow[d, "\Phi_K(\zeta^1)"'] & (t_2)_*t_2^*\Phi_K \arrow[d, "(t_2)_*(\beta^K)"] \\
\Phi_K(t_1)_*t_1^* \arrow[r, "\alpha^K_{t_1^*}"] & (t_2)_*\Phi_{(t')^*K}t_1^*.
\end{tikzcd} \tag{7.9}$$

Proposition 7.7. *Consider Setup 6.1 where t is affine. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over Y_2. Denote by K' for the kernel of the integral transform obtained in Proposition 6.12 which is right adjoint to Φ_K on D_{qc}. Then, after possibly replacing $\Phi_{\mathbf{L}(t')^*K}$ by an isomorphic functor, we have a morphism of adjoint pairs:*

$$\begin{tikzcd}
D_{\mathrm{qc}}(Y_1) \arrow[r, "\mathbf{L}t_1^*"] \arrow[d, shift right=2, "\Phi_K"'] \arrow[d, phantom, "\dashv"] & D_{\mathrm{qc}}(Y_1 \times_S T) \arrow[d, shift right=2, "\Phi_{\mathbf{L}(t')^*K}"'] \arrow[d, phantom, "\dashv"] \\
D_{\mathrm{qc}}(Y_2) \arrow[r, "\mathbf{L}t_2^*"'] \arrow[u, shift right=2, "\Phi_{K'}"'] & D_{\mathrm{qc}}(Y_2 \times_S T) \arrow[u, shift right=2, "\Phi_{\mathbf{L}(t')^*K'}"']
\end{tikzcd}$$

with left and right comparison transformations given respectively by the isomorphisms

$$(\beta^K)^{-1} \colon \Phi_{\mathbf{L}(t')^*K} \circ \mathbf{L}t_1^* \to \mathbf{L}t_2^* \circ \Phi_K,$$
$$\beta^{K'} \colon \mathbf{L}t_1^* \circ \Phi_{K'} \to \Phi_{\mathbf{L}(t')^*K'} \circ \mathbf{L}t_2^*.$$

Proof. Write η (resp. η') for the unit of the adjunction $\Phi_K \dashv \Phi_{K'}$ (resp. of $\Phi_{(t')^*K} \dashv \Phi_{(t')^*K'}$). By Lemma 7.1(2), proving the Proposition amounts to showing the following diagram commutes:

$$\begin{tikzcd}
\mathbf{L}t_1^* \arrow[r, "\eta'_{\mathbf{L}t_1^*}"] \arrow[d, "\mathbf{L}t_1^*(\eta)"'] & \Phi_{\mathbf{L}(t')^*K'}\Phi_{\mathbf{L}(t')^*K}\mathbf{L}t_1^* \\
\mathbf{L}t_1^*\Phi_{K'}\Phi_K \arrow[r, "\beta^{K'}_{\Phi_K}"'] & \Phi_{\mathbf{L}(t')^*K'}\mathbf{L}t_2^*\Phi_K. \arrow[u, "\Phi_{\mathbf{L}(t')^*K'}(\beta^K)"']
\end{tikzcd} \tag{7.10}$$

As in the proof of Lemma 7.5, we omit $\mathbf{L}$'s and $\mathbf{R}$'s in the notation for the derived functors to ease notation. Now, the bad news is that (7.10) needs not commute. Although, the good news is that it does commute if we replace $\Phi_{(t')^*K}$ by some appropriate isomorphic functor.

Specifically, we claim there is an adjunction $\widetilde{\Phi}_{(t')^*K} \dashv \Phi_{(t')^*K'}$ with unit $\tilde{\eta}'$ making (7.10) commute (i.e. inscribing $\widetilde{(-)}$ to $\Phi_{(t')^*K}$ and η') and that moreover there is an isomorphism $\theta : \Phi_{(t')^*K} \cong \widetilde{\Phi}_{(t')^*K}$ satisfying

$$\Phi_{(t')^*K'}(\theta) \circ \eta' = \tilde{\eta}'. \tag{7.11}$$

Assume that for every $E \in D_{\mathrm{qc}}(Y_1)$ we have shown:

($\mathscr{P}(E)$) The object $\Phi_{(t')^*K} t_1^* E$ with the morphism $\Phi_{(t')^*K}(\beta^K_E) \circ \beta^{K'}_{\Phi_K E} \circ t_1^*(\eta_E)$ is a reflection of t_1^*E along $\Phi_{(t')^*K'}$ (in the sense of [Bor94, Definition 3.1.1]).

Then we can define $\widetilde{\Phi}_{(t')^*K}$ by means of [Bor94, Proposition 3.1.3]. On the one hand, we have $\Phi_{(t')^*K} \dashv \Phi_{(t')^*K'}$ with unit η'; thus by [Bor94, Theorem 3.1.5] the pair $(\Phi_{(t')^*K}A, \eta'_A)$ is a reflection of $A \in D_{\mathrm{qc}}(Y_1 \times_S T)$ along $\Phi_{(t')^*K'}$. Now define $\tilde{\eta}' = \{\tilde{\eta}'_A\}_{A \in D_{\mathrm{qc}}(Y_1 \times_S T)}$ by setting $\tilde{\eta}'_A := \eta'_A$ if A is not in the image of $t_1^* : D_{\mathrm{qc}}(Y_1) \to D_{\mathrm{qc}}(Y_1 \times_S T)$ and setting $\tilde{\eta}'_{t_1^*E}$, $E \in D_{\mathrm{qc}}(Y_1)$, to be the unique top morphism in (7.10) making it commute. By [Bor94, Proposition 3.1.3], there is a unique functor $\widetilde{\Phi}_{(t')^*K} : D_{\mathrm{qc}}(Y_1 \times_S T) \to D_{\mathrm{qc}}(Y_2 \times_S T)$ whose action on objects coincides with $\Phi_{(t')^*K}$ and such that $\tilde{\eta}' : \mathrm{id} \to \widetilde{\Phi}_{(t')^*K'} \Phi_{(t')^*K}$ is a natural transformation. Moreover, by [Bor94, Theorem 3.1.5], we have $\widetilde{\Phi}_{(t')^*K} \dashv \Phi_{(t')^*K'}$ with unit $\tilde{\eta}'$. Finally, by [Rie17, Proposition 4.4.1], there is a natural isomorphism $\theta : \Phi_{(t')^*K} \cong \widetilde{\Phi}_{(t')^*K}$ such that (7.11) holds.

It is left for us to show $\mathscr{P}(E)$. So, let $A \in D_{\mathrm{qc}}(Y_2)$ and consider some $\gamma \colon t_1^*E \to \Phi_{(t')^*K'}A$. Our goal is to check that there is a unique morphism $\tilde{\gamma} \colon \Phi_{(t')^*K} t_1^* E \to A$ such that the diagram

$$\begin{tikzcd}
t_1^*E \arrow[d, "t_1^*(\eta_E)"'] \arrow[ddr, "\gamma"] & \\
t_1^*\Phi_{K'}\Phi_K E \arrow[d, "\beta^{K'}_{\Phi_K(E)}"'] & \\
\Phi_{(t')^*K'} t_2^* \Phi_K E \arrow[d, "\Phi_{(t')^*K'}(\beta^K_E)"'] & \Phi_{(t')^*K'}A \\
\Phi_{(t')^*K'}\Phi_{(t')^*K} t_1^* E \arrow[ur, "\Phi_{(t')^*K'}(\tilde{\gamma})"'] &
\end{tikzcd} \tag{7.12}$$

commutes. Let $i \in \{1,2\}$. Denote ζ^i to the unit of $t_i^* \dashv (t_i)_*$. Given morphisms

$$\phi \colon t_i^* M \to B, \quad \psi \colon C \to (t_i)_* D, \quad \rho \colon L \to \Phi_{K'} P,$$

we will use notations

$$\phi^{\flat_i} \colon M \to (t_i)_* B, \quad \psi^{\sharp_i} \colon t_i^* C \to D, \quad \rho^{\sharp} \colon \Phi_K L \to P$$

for the adjunct morphisms of ϕ, ψ, ρ with respect to the adjunctions $t_i^* \dashv (t_i)_*$ and $\Phi_K \dashv \Phi_{K'}$. Suppose there is $\tilde{\gamma}$ turning (7.12) into a commutative diagram. Then $\gamma^{\flat_1} \colon E \to$

$(t_1)_*\Phi_{(t')^*K'}A$ equals the outer clockwise composite in the following diagram:

$$\begin{tikzcd}
E \arrow[rr, "\zeta^1_E"] \arrow[d, "\eta_E"] \arrow[dddd, bend right=60, "\delta"'] & & (t_1)_*t_1^*E \arrow[d, "(t_1)_*t_1^*(\eta_E)"] \\
\Phi_{K'}\Phi_K E \arrow[rr, "\zeta^1_{\Phi_{K'}\Phi_K E}"] \arrow[d, "\Phi_{K'}(\zeta^2_{\Phi_K E})"'] & & (t_1)_*t_1^*\Phi_{K'}\Phi_K E \arrow[d, "(t_1)_*(\beta^{K'}_{\Phi_K(E)})"] \\
\Phi_{K'}(t_2)_*t_2^*\Phi_K E \arrow[rr, "\alpha^{K'}_{t_2^*\Phi_K E}"] \arrow[d, "\Phi_{K'}(t_2)_*(\beta^K_E)"] & & (t_1)_*\Phi_{(t')^*K'}t_2^*\Phi_K E \arrow[d, "(t_1)_*\Phi_{(t')^*K'}(\beta^K_E)"] \\
\Phi_{K'}(t_2)_*\Phi_{(t')^*K}t_1^*E \arrow[d, "\Phi_{K'}(t_2)_*(\tilde{\gamma})"] & & (t_1)_*\Phi_{(t')^*K'}\Phi_{(t')^*K}t_1^*E \arrow[d, "(t_1)_*\Phi_{(t')^*K'}(\tilde{\gamma})"] \\
\Phi_{K'}(t_2)_*A \arrow[rr, "\alpha^{K'}_A"'] & & (t_1)_*\Phi_{(t')^*K'}A.
\end{tikzcd}$$

In this diagram, we note the upper square commutes by naturality of ζ^1, the middle square commutes by Lemma 7.5 (it is (7.9) with K replaced by K'), and the bottom rectangle commutes by naturality of $\alpha^{K'}$. Set δ for the left vertical composite in the last diagram, i.e. $\delta = (\alpha^{K'}_A)^{-1} \circ \gamma^{\flat_1} \colon E \to \Phi_{K'}(t_2)_*A$. Then it follows that

$$\delta^\sharp \colon \Phi_K E \xrightarrow{\zeta^2_{\Phi_K E}} (t_2)_*t_2^*\Phi_K E \xrightarrow{(t_2)_*(\beta^K_E)} (t_2)_*\Phi_{(t')^*K}t_1^*E \xrightarrow{(t_2)_*(\tilde{\gamma})} (t_2)_*A,$$

which tells us

$$(\delta^\sharp)^{\sharp_2} \colon t_2^*\Phi_K E \xrightarrow{\beta^K_E} \Phi_{(t')^*K}t_1^*E \xrightarrow{\tilde{\gamma}} A.$$

Consequently, we have

$$\begin{aligned}
\tilde{\gamma} &= (\delta^\sharp)^{\sharp_2} \circ (\beta^K_E)^{-1} \\
&= ([(\alpha^{K'}_A)^{-1} \circ \gamma^{\flat_1}]^\sharp)^{\sharp_2} \circ (\beta^K_E)^{-1},
\end{aligned}$$

which shows uniqueness of $\tilde{\gamma}$. Conversely, defining $\tilde{\gamma}$ by the last formula and reading the previous derivations in reverse order, it follows that (7.12) commutes. □

Remark 7.8. Consider the situation as in Proposition 7.7. Denote ε and ε' respectively for the counits of the adjunctions $\Phi_K \dashv \Phi_{K'}$ and $\Phi_{\mathbf{L}(t')^*K} \dashv \Phi_{\mathbf{L}(t')^*K'}$. From Lemma 7.1, it follows that (after possibly replacing $\Phi_{\mathbf{L}(t')^*K}$ by an isomorphic functor) the following diagram commutes:

$$\begin{tikzcd}
\Phi_{\mathbf{L}(t')^*K}\mathbf{L}t_1^*\Phi_{K'} \arrow[d, "\Phi_{\mathbf{L}(t')^*K}(\beta^{K'})"'] & \mathbf{L}t_2^*\Phi_K\Phi_{K'} \arrow[l, "\beta^K_{\Phi_{K'}}"'] \arrow[d, "\mathbf{L}t_2^*(\varepsilon)"] \\
\Phi_{\mathbf{L}(t')^*K}\Phi_{\mathbf{L}(t')^*K'}\mathbf{L}t_2^* \arrow[r, "\varepsilon'_{\mathbf{L}t_2^*}"] & \mathbf{L}t_2^*.
\end{tikzcd} \tag{7.13}$$

Also, the proof of $\mathscr{P}(E)$ never used that t is affine. Specifically, we showed that with Setup 6.1 and given $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ relatively perfect over Y_2, it is the case that $\mathscr{P}(E)$

holds for $E \in D_{\mathrm{qc}}(Y_1)$.[2] Thus, if it also happens that $t_1^* \colon D_{\mathrm{qc}}(Y_1) \to D_{\mathrm{qc}}(Y_1 \times_S T)$ is essentially surjective, then $\Phi_{(t')^*K'} \colon D_{\mathrm{qc}}(Y_2 \times_S T) \to D_{\mathrm{qc}}(Y_1 \times_S T)$ has a left adjoint L (see e.g. [Bor94, Theorem 3.1.5]) such that there is an isomorphism

$$L(A) \cong \Phi_{(t')^*K}(A)$$

for each $A \in D_{\mathrm{qc}}(Y_1 \times_S T)$. However, one should be careful, for the isomorphism above might *not* be natural in A.

Lemma 7.9. *Consider a faithfully flat morphism $f \colon Y \to X$ of schemes. Then $\mathbf{L}f^* \colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ is conservative.*

Proof. Let $E \in D_{\mathrm{qc}(X)}$ satisfy $\mathbf{L}f^*E \cong 0$. Since f is flat, we have $f^*\mathcal{H}^j(E) \cong \mathcal{H}^j(\mathbf{L}f^*E)$ for all $j \in \mathbb{Z}$. Thus, it suffices to check that $f^*\mathcal{H}^j(E) \cong 0$ implies $\mathcal{H}^j(E) \cong 0$ for all $j \in \mathbb{Z}$. This follows from [GW20, Proposition 14.11]. □

Lemma 7.10. *Let $F \colon \mathcal{T} \leftrightarrows \mathcal{S} \colon G$ be a pair of exact adjoint functors between compactly generated triangulated categories. Assume G preserves small coproducts. Then F is an equivalence if, and only if, F restricts to an equivalence $\mathcal{T}^c \to \mathcal{S}^c$. A similar statement holds for F being fully faithful. Also, if F restricts to a fully faithful functor $\mathcal{T}^c \to \mathcal{S}^c$ and $F(\mathcal{T}^c)$ compactly generates $\mathcal{S}$, then $F \colon \mathcal{S} \to \mathcal{T}$ is an equivalence. Additionally, if F restricts to a fully faithful functor $\mathcal{T}^c \to \mathcal{S}^c$, then F is fully faithful $\mathcal{T} \to \mathcal{S}$*

Proof. It is straightforward to check that F being an equivalence induces an equivalence $\mathcal{T}^c \to \mathcal{S}^c$. So, we check the converse. To do so, we show that F and G are both fully faithful. Denote the counit and unit of the adjunction respectively by ϵ and η. Set $\mathcal{T}'$ and $\mathcal{S}'$ the strictly full subcategories respectively of $\mathcal{T}$ and $\mathcal{S}$ consisting of objects where ϵ and η are isomorphisms. One may verify that $\mathcal{T}'$ and $\mathcal{S}'$ are closed under shifts, iterated cones, and small coproducts (hence, homotopy colimits). However, [Sta26, Tag 09SN] tells us each object in $\mathcal{T}$ and $\mathcal{S}$ can be obtained respectively by $\mathcal{T}^c$ and $\mathcal{S}^c$ using these operations. Hence, $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{S} \subseteq \mathcal{S}'$, which completes the proof of the first claim.

The remaining claims follow similarly. Indeed, we show the very last claim. Let $\mathcal{T}'$ denote the strictly full subcategory of $E \in \mathcal{T}$ such that $\eta_E \colon E \to GF(E)$ is an isomorphism. It can be checked that $\mathcal{T}'$ is a triangulated subcategory of $\mathcal{T}$ which is closed under small coproducts. By hypothesis, it follows that $\mathcal{T}^c \subseteq \mathcal{T}'$. Since $\mathcal{T}$ is compactly generated, it follows that $\mathcal{T} = \mathcal{T}'$. Therefore, η_E is an isomorphism for al $E \in \mathcal{T}$. □

Proposition 7.11. *Consider Setup 6.1 where t is affine. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over each Y_i. If Φ_K is fully faithful (resp. an equivalence) on D^b_{coh}, then so is $\Phi_{\mathbf{L}(t')^*K}$.*

Proof. We only prove the case of fully faithfulness since the other is analagous. By Theorem 6.10, $\mathbf{L}(t')^*K$ is relatively perfect over each $Y_i \times_S T$. By Proposition 6.12, Φ_K and $\Phi_{K'}$, as well as $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$, form adjoint pairs on D_{qc}. Denote by η (resp. η') the unit of the adjoint pair Φ_K and $\Phi_{K'}$ (resp. $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$) on D_{qc}. By Corollary 6.13, these adjoint pairs restrict to D^b_{coh}. Moreover, [AJS23, Theorem 2.3] implies that Φ_K and $\Phi_{\mathbf{L}(t')^*K}$ restrict to functors on Perf. Let $G \in D_{\mathrm{qc}}(Y_1)$ be a compact generator for $D_{\mathrm{qc}}(Y_1)$.

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[2] The proof of Proposition 7.7 only needs t affine to guarantee that $\Phi_{(t')^*K} \dashv \Phi_{(t')^*K'}$, so that $\mathcal{P}(E)$ plus a replacement of $\Phi_{(t')^*K}$ by a certain isomorphic functor yields a morphism of adjunctions as in the statement of Proposition 7.7.

By [HR17, Corollary 2.8] and [Lan26, Corollary 3.6], $\mathbf{L}t_1^*G$ is a compact generator for $D_{\mathrm{qc}}(Y_1 \times_S T)$. By (7.10), we obtain

$$\operatorname{cone}(\eta'_{\mathbf{L}t_1^*G}) \cong \mathbf{L}t_1^* \operatorname{cone}(\eta_G).$$

Since Φ_K is fully faithful on Perf, we have that $\operatorname{cone}(\eta'_{\mathbf{L}t_1^*G}) \cong 0$. Let $\mathscr{A}$ be the collection of objects E in $D_{\mathrm{qc}}(Y_1 \times_S T)$ such that η'_E is an isomorphism. We have shown that $\mathbf{L}t_1^*G \in \mathscr{A}$. Therefore, Lemma 7.10 implies $\mathscr{A} = D_{\mathrm{qc}}(Y_1 \times_S T)$, which completes the proof. □

Proposition 7.12. *Consider Setup 6.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over each Y_i. If $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful (resp. an equivalence) on D^b_{coh} where t is one of the following:*

1. *t is affine and faithfully flat (e.g. $t = 1_S$)*
2. *$t\colon \operatorname{Spec}(k) \to S$ is any morphism from a field (so here, we want the condition to all for all such morphisms)*
3. *$t\colon \operatorname{Spec}(\kappa(p)) \to S$ is any closed immersion associated to a closed point $p \in S$ (so here, we want the condition to all for all such morphisms),*

then so is Φ_K.

Proof. Set K' to be the kernel of the integral transform obtained in Proposition 6.12 which is right adjoint to Φ_K on D_{qc}. In both cases, we prove the claim for full faithfulness, as the remaining assertion is argued similarly. We first recall some properties satisfied by any morphism t appearing in the statement. By Theorem 6.10, $\mathbf{L}(t')^*K$ is relatively perfect over each $Y_i \times_S T$. By Proposition 6.12, Φ_K and $\Phi_{K'}$, as well as $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$, form adjoint pairs on D_{qc}. Denote by η (resp. η') the unit of the adjunction Φ_K and $\Phi_{K'}$ (resp. $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$) on D_{qc}. By Corollary 6.13, these adjoint pairs restrict to D^b_{coh}. Moreover, Theorem 6.16 implies that Φ_K and $\Phi_{\mathbf{L}(t')^*K}$ restrict to functors on Perf.

We start by proving (7.12(1)). Let $E \in D^b_{\mathrm{coh}}(Y_1)$. Since t is faithfully flat, t_1 is as well, and so $\mathbf{L}t_1^*E \in D^b_{\mathrm{coh}}(Y_1 \times_S T)$. By (7.10), it follows that

$$\operatorname{cone}(\eta'_{\mathbf{L}t_1^*E}) \cong \mathbf{L}t_1^* \operatorname{cone}(\eta_E).$$

However, $\Phi_{\mathbf{L}(t')^*K}$ restricts to a fully faithful functor on D^b_{coh}, and so $\operatorname{cone}(\eta'_{\mathbf{L}t_1^*E}) \cong 0$. Then Lemma 7.9 implies that $\operatorname{cone}(\eta_E) \cong 0$. Hence, Φ_K restricts to a fully faithful functor on D^b_{coh} because E was arbitrary.

Lastly we check (7.12(3)). By Lemma 7.10, it suffices to check Φ_K restricts to a fully faithful functor on $D_{\mathrm{qc}}(Y_1)^c$. Note that $D_{\mathrm{qc}}(Y_1)^c \subseteq \operatorname{Perf}(Y_1)$. Choose any $P \in D_{\mathrm{qc}}(Y_1)^c$. Since Φ_K restricts to a functor on Perf, we know that $\eta_P \in \operatorname{Perf}(Y_1)$. Let $p \in |Y_1|$ be a closed point. Suppose $s\colon \operatorname{Spec}(k) \to Y_1$ is the natural morphism where $k := \kappa(p)$. Since f_1 is proper, $f_1 \circ s$ is a proper morphism. Hence, there exists an induced field extension $k/\kappa(t(p))$. Set t to be the associated closed immersion of $\kappa(t(p))$.

Consider the following commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[rrd, bend left, "\widetilde{t}"] \arrow[rd, "h"] \arrow[rdd, bend right, "s"'] & & \\
& Y_1 \times_S \operatorname{Spec}(k) \arrow[r, "f_1'"] \arrow[d, "t_1"] & \operatorname{Spec}(\kappa(t(p))) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"'] & S.
\end{tikzcd}$$

By (7.10), we obtain that

$$\operatorname{cone}(\eta'_{\mathbf{L}t_1^*P}) \cong \mathbf{L}t_1^* \operatorname{cone}(\eta_P).$$

Since P is perfect, we know that $\mathbf{L}t_1^*P \in \operatorname{Perf}(Y_1 \times_S \operatorname{Spec}(k))$. By hypothesis, $\Phi_{\mathbf{L}(t')^*K}$ is fully faithful, and so $\operatorname{cone}(\eta'_{\mathbf{L}t_1^*P}) \cong 0$. Then

$$\mathbf{L}s^* \operatorname{cone}(\eta_P) \cong \mathbf{L}h^*\mathbf{L}t_1^* \operatorname{cone}(\eta_P) \cong 0.$$

By Corollary 2.13, we see that $p \notin \operatorname{supp}(\operatorname{cone}(\eta_P))$. Since p was an arbitrary closed point of Y_1, it follows that $\operatorname{supp}(\operatorname{cone}(\eta_P)) = \emptyset$. In other words, $\operatorname{cone}(\eta_P)$ is the zero object, which completes the proof. □

Proposition 7.13. *Consider Setup 6.1 where t is affine. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over Y_2. Denote by K' for the kernel of the integral transform obtained in Proposition 6.12 which is right adjoint to Φ_K on D_{qc}. Then, after possibly replacing $\Phi_{K'}$ by an isomorphic functor, we have a morphism of adjoint pairs as depicted in the diagram*

$$\begin{tikzcd}
D_{\mathrm{qc}}(Y_1 \times_S T) \arrow[r, "\mathbf{R}(t_1)_*"] \arrow[d, shift right=2, "\Phi_{\mathbf{L}(t')^*K}"'] \arrow[d, phantom, "\dashv"] & D_{\mathrm{qc}}(Y_1) \arrow[d, shift right=2, "\Phi_K"'] \arrow[d, phantom, "\dashv"] \\
D_{\mathrm{qc}}(Y_2 \times_S T) \arrow[r, "\mathbf{R}(t_2)_*"'] \arrow[u, shift right=2, "\Phi_{\mathbf{L}(t')^*K'}"'] & D_{\mathrm{qc}}(Y_2) \arrow[u, shift right=2, "\Phi_{K'}"']
\end{tikzcd}$$

with left and right comparison transformations given respectively by the isomorphisms

$$\alpha^K \colon \Phi_K \circ \mathbf{R}(t_1)_* \to \mathbf{R}(t_2)_* \circ \Phi_{\mathbf{L}(t')^*K},$$
$$(\alpha^{K'})^{-1} \colon \mathbf{R}(t_1)_* \circ \Phi_{\mathbf{L}(t')^*K'} \to \Phi_{K'} \circ \mathbf{R}(t_2)^*.$$

Proof. This argument is essentially dual to that of Proposition 7.7. □

Remark 7.14. Consider the situation as in Proposition 7.13. Denote ε and ε' respectively for the counits of the adjunctions $\Phi_K \dashv \Phi_{K'}$ and $\Phi_{\mathbf{L}(t')^*K} \dashv \Phi_{\mathbf{L}(t')^*K'}$. From Lemma 7.1, it follows that (after possibly replacing $\Phi_{K'}$ by an isomorphic functor) the following diagrams commute:

(7.14)

$$\begin{tikzcd}
\Phi_K \mathbf{L}(t_1)_* \Phi_{\mathbf{L}(t')^*K'} \arrow[r, "\alpha^K_{\Phi_{\mathbf{L}(t')^*K'}}"] & \mathbf{R}(t_2)_* \Phi_{\mathbf{L}(t')^*K} \Phi_{\mathbf{L}(t')^*K'} \arrow[d, "\mathbf{R}(t_2)_*(\varepsilon')"] \\
\Phi_K \Phi_{K'} \mathbf{R}(t_2)_* \arrow[u, "\Phi_K(\alpha^{K'})"] \arrow[r, "\varepsilon_{\mathbf{R}(t_2)_*}"'] & \mathbf{R}(t_2)_*
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{R}(t_1)_\ast \arrow[r, "\eta_{\mathbf{R}(t_1)_\ast}"] \arrow[d, "\mathbf{R}(t_1)_\ast(\eta')"'] & \Phi_{K'}\Phi_K\mathbf{R}(t_1)_\ast \arrow[d, "\Phi_{K'}(\alpha^K)"] \\
\mathbf{R}(t_1)_\ast\Phi_{\mathbf{L}(t')^\ast K'}\Phi_{\mathbf{L}(t')^\ast K} & \Phi_{K'}\mathbf{R}(t_2)_\ast\Phi_{\mathbf{L}(t')^\ast K}. \arrow[l, "\alpha^{K'}_{\Phi_{\mathbf{L}(t')^\ast K}}"]
\end{tikzcd} \tag{7.15}$$

Theorem 7.15. *Consider Setup 6.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over each Y_i. Then the following are equivalent:*

1. *Φ_K is fully faithful (resp. an equivalence) on D^b_{coh}*
2. *$\Phi_{\mathbf{L}(t')^\ast K}$ is fully faithful (resp. an equivalence) on D^b_{coh} for any affine morphism t*
3. *$\Phi_{\mathbf{L}(t')^\ast K}$ is fully faithful (resp. an equivalence) on D^b_{coh} for every morphism $t\colon \operatorname{Spec}(k) \to S$ from a field*
4. *$\Phi_{\mathbf{L}(t')^\ast K}$ is fully faithful (resp. an equivalence) on D^b_{coh} for every closed point $p \in S$ with associated closed immersion $t\colon \operatorname{Spec}(\kappa(p)) \to S$*
5. *$\Phi_{\mathbf{L}(t')^\ast K}$ is fully faithful (resp. an equivalence) on D^b_{coh} for some affine surjection t.*

Proof. (1) $\implies$ (2) follows from Proposition 7.11; it is straight to check that (2) $\implies$ (3) because Noetherian schemes are affine-pointed; (3) $\implies$ (4) is easy to show; (4) $\implies$ (1) follows from Proposition 7.12. □

Corollary 7.16 (cf. [Orl02, Lemma 2.12]). *Consider proper schemes Y_1 and Y_2 over a field k. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_k Y_2)$ be relatively perfect over each Y_i. Then Φ_{K_L} is fully faithful (resp. an equivalence) on D^b_{coh} for any field extension L/k if, and only if, Φ_K is fully faithful (resp. an equivalence) on D^b_{coh}.*

Remark 7.17. Notably, the techniques we use are independent of those appearing in [Orl02, Lemma 2.12].

8. Derived invariances

In this section, we study the consequences of Theorem 7.15. By 'genus' of a curve, we mean 'arithmetic'(see e.g. [Sta26, Tag 0BY6]).

Proposition 8.1. *Let k be a field of characteristic zero. Suppose C and C' are geometrically integral projective curves over k. If C and C' are Fourier–Mukai partners over k, then $\operatorname{Spec}(L) \times_k C \cong \operatorname{Spec}(L) \times_k C'$ for some finite field extension L/k (i.e. are k-forms).*

Proof. Let $\overline{k}$ be the algebraic closure of k. Suppose $K \in D^b_{\mathrm{coh}}(C \times_k C')$ has an associated integral transform Φ_K restricting to an equivalence $D^b_{\mathrm{coh}}(C) \to D^b_{\mathrm{coh}}(C')$. Denote by $\pi\colon (C \times_k C') \times_k \operatorname{Spec}(\overline{k}) \to C \times_k C'$ the natural morphism. From Theorem 7.15, we see that $\Phi_{\mathbf{L}\pi^\ast K}$ restricts to an equivalence $D^b_{\mathrm{coh}}(C \times_k \operatorname{Spec}(\overline{k})) \to D^b_{\mathrm{coh}}(C' \times_k \operatorname{Spec}(\overline{k}))$. Then $C \times_k \operatorname{Spec}(\overline{k})$ and $C' \times_k \operatorname{Spec}(\overline{k})$ are Fourier–Mukai partners over an algebraically closed field of characteristic zero. By [CNS25b, Corollary 5.4], we have that $\Phi_{\mathbf{L}\pi^\ast K}$ restricts to an equivalence $\operatorname{Perf}(C \times_k \operatorname{Spec}(\overline{k})) \to \operatorname{Perf}(C' \times_k \operatorname{Spec}(\overline{k}))$. Consequently, we know that $C \times_k \operatorname{Spec}(\overline{k}) \cong C' \times_k \operatorname{Spec}(\overline{k})$ in the following cases:

- $C \times_k \operatorname{Spec}(\overline{k})$ is Gorenstein with ample or anti-ample canonical bundle [Bal11, Theorem 2]

- $C \times_k \operatorname{Spec}(\overline{k})$ is not Gorenstein [Spe23, Theorem A]
- $C \times_k \operatorname{Spec}(\overline{k})$ Gorenstein of genus one and trivial canonical line bundle [LM14, Theorem 1.1] ($C \times_k \operatorname{Spec}(\overline{k})$ is integral with genus one such that $\deg(\omega_X) = 0$ implies $\omega_X \cong \mathcal{O}_X$ by [Cat82, Proposition 1.11] where ω_X is the canonical sheaf).

Observe these are the only possible cases for projective curves over an algebraically closed field in characteristic zero (see e.g. the two paragraphs above [Spe23, Theorem A] or use [Cat82, Theorem B, Proposition 1.8 and Proposition 1.11]). Hence, there is a finite field extension L/k such that $C \times_k \operatorname{Spec}(L) \cong C' \times_k \operatorname{Spec}(L)$ (see e.g. [Poo17, Proposition 3.2.5]), which completes the proof. □

Corollary 8.2. *Let S be a Noetherian $\mathbb{Q}$-scheme. Suppose $f_i \colon Y_i \to S$ are proper flat morphisms whose special fibers are geometrically integral curves. If Y_1 and Y_2 are Fourier–Mukai partners over S, then then genii of the special fibers of Y_1 coincide with that of Y_2.*

Proof. Let $s \in S$ be a closed point. Then the natural morphism

$$\operatorname{Spec}(\kappa(s)) \to S \to \operatorname{Spec}(\mathbb{Q})$$

implies that $\mathbb{Q} \subseteq \kappa(s)$ (i.e. $\kappa(s)$ has characteristic zero). Note that $\operatorname{Spec}(\kappa(s)) \to S$ is affine (see e.g. [Sta26, Tag 01SI]). Denote by Y_i^s for the base change of Y_i along $\operatorname{Spec}(\kappa(s)) \to S$. Our hypothesis ensures that each Y_i^s is a geometrically integral projective curve over $\kappa(s)$ (see e.g. [Sta26, Tag 0A26]). Then, by Theorem 7.15, Y_1^s and Y_2^s are Fourier–Mukai partners over $\kappa(s)$. Furthermore, Proposition 8.1 guarantees that there is a finite field extension $K(s)/\kappa(s)$ such that $Y_1^s \times_{\kappa(s)} \operatorname{Spec}(K(s))$ is isomorphism to $Y_2^s \times_{\kappa(s)} \operatorname{Spec}(K(s))$. Hence, if coupled with [Sta26, Tag 0BY9 & Tag 0FD2], the genii of each $Y_i^s \times_{\kappa(s)} \operatorname{Spec}(K(s))$ coincides with Y_j^s for $i \neq j$. In other words, the genii of Y_1^s and Y_2^s coincide, which completes the proof. □

Lemma 8.3. *Let Y_1 and Y_2 be proper schemes over a field k (which need not be perfect). Suppose Y_1 and Y_2 are Fourier–Mukai partners. If Y_1 is geometrically regular, then so is Y_2. In other words, smoothness is an invariance for Fourier-Mukai partners.*

Proof. By Theorem 7.15, $Y_1 \times_k \operatorname{Spec}(L)$ and $Y_2 \times_k \operatorname{Spec}(L)$ are Fourier–Mukai partners for every finitely generated field extension L/k. So, $Y_1 \times_k \operatorname{Spec}(L)$ is regular. Then [CNS25b, Corollary 5.4] implies the following string of triangulated equivalences:

$$D^b_{\operatorname{coh}}(Y_1 \times_k \operatorname{Spec}(L)) \cong \operatorname{Perf}(Y_1 \times_k \operatorname{Spec}(L)) \cong \operatorname{Perf}(Y_2 \times_k \operatorname{Spec}(L)).$$

Hence, $\operatorname{Perf}(Y_2 \times_k \operatorname{Spec}(L)) = D^b_{\operatorname{coh}}(Y_2 \times_k \operatorname{Spec}(L))$, telling us that $Y_2 \times_k \operatorname{Spec}(L)$ is regular. In other words, Y_2 must be geometrically regular. □

Proposition 8.4. *Let $f_i \colon Y_i \to S$ be proper flat morphisms to a Noetherian scheme. Assume that Y_1 and Y_2 are Fourier–Mukai partners over S. Then f_1 is smooth if, and only if, f_2 is smooth.*

Proof. We prove only one direction as the other follows nearly verbatim. Assume f_1 is smooth. It suffices, by [Sta26, Tag 01V8], to show that all the fibers $Y_2 \times_S \operatorname{Spec}(\kappa(s))$ are smooth over $\kappa(s)$ where $s \in S$ is any point. Fix a point $s \in S$ (which need not be closed). The natural morphism $t_s \colon \operatorname{Spec}(\mathcal{O}_{S,s}) \to S$ is affine (see e.g. [Sta26, Tag 01SI]). Hence, Theorem 7.15 tells us the base changes of Y_1 and Y_2 along t_s are Fourier–Mukai partners. A further application Theorem 7.15 implies these base changes remain Fourier–Mukai partners if we base change further along the natural closed immersion

$\mathrm{Spec}(\kappa(p)) \to \mathrm{Spec}(\mathcal{O}_{S,s})$. However, smoothness is stable under base change, so each base change of Y_1 along such morphisms remains smooth. Hence, the desired claim follows from Lemma 8.3. □

Proposition 8.5. *Let $f_i \colon Y_i \to S$ be projective flat morphisms with geometrically integral fibers to a Noetherian scheme. Assume that Y_1 and Y_2 are Fourier–Mukai partners over S. Then f_1 is Cohen–Macaulay (resp. Gorenstein) if, and only if, f_2 satisfies the same condition.*

Proof. It suffices to show f_2 is Cohen–Macaulay (resp. Gorenstein) if f_1 satisfies the same condition because the proof is analogous for the converse. In fact, we argue in a similar fashion to Proposition 8.4. Moreover, by [Sta26, Tag 045O] (resp. [Sta26, Tag 0C03]), we only need to show that all the fibers $Y_2 \times_S \mathrm{Spec}(\kappa(s))$ are Cohen–Macaulay (resp. Gorenstein) over an algebraic closure of each $\kappa(s)$ where $s \in S$ is any point. Fix $s \in S$ (perhaps, not closed). The natural morphism $t_s \colon \mathrm{Spec}(\mathcal{O}_{S,s}) \to S$ is affine (see e.g. [Sta26, Tag 01SG]). Hence, Theorem 7.15 tells us the base changes of Y_1 and Y_2 along t_s are Fourier–Mukai partners. Then Theorem 7.15 implies these base changes remain Fourier–Mukai partners if we base change further along the natural closed immersion $\mathrm{Spec}(\kappa(p)) \to \mathrm{Spec}(\mathcal{O}_{S,s})$. Furthermore, if we base change further along $\overline{\kappa(s)}/\kappa(s)$ with $\overline{\kappa(s)}$ an algebraic closure of $\kappa(s)$, these base changes remain Fourier–Mukai partners via Theorem 7.15. However, it follows (as fibers are geometrically integral) from [HCS09, Theorem 4.4], that these integral projective schemes over $\overline{\kappa(s)}$ enjoy the property that Cohen–Macaulayness (resp. Gorensteinnes) is a derived invariance. This completes the proof. □

Proposition 8.6. *Let $f_i \colon Y_i \to S$ be projective flat morphisms to a separated Noetherian scheme, where Y_1 and Y_2 are of finite Krull dimension, satisfying the resolution property (see e.g. [Sta26, Tag 0F86]). Assume that Y_1 and Y_2 are derived equivalent (e.g. the equivalence need not arise from an integral transform). Then f_1 is Gorenstein if, and only if, f_2 satisfies the same condition.*

Proof. We prove the case where f_1 is Gorenstein implies that of f_2 because the proof for the converse holds verabtim. Denote by $D^b_{\mathrm{fid}}(Y_2)$ the full triangulated subcategory in $D^b_{\mathrm{coh}}(Y_2)$ of complexes with bounded cohomology and with a bounded injective resolution (i.e. can be represented by a complex with finitely many nonzero components where each is injective $\mathcal{O}_X$-module). By [Har66, Subchapter V.2], the dualizing sheaf $f_2^!\mathcal{O}_S$ is contained in $D^b_{\mathrm{fid}}(Y_2)$. Furthermore, [Sta26, Tag 0A89] tells us that there is an equivalence

$$D := \mathbf{R}\mathcal{H}om(-, f_2^!\mathcal{O}_S) \colon D^b_{\mathrm{coh}}(Y_2) \xrightarrow{\sim} D^b_{\mathrm{coh}}(Y_2)^{\mathrm{op}} \tag{8.1}$$

satisfying $D \circ D = \mathrm{id}$. Moreover, this equivalence restricts to an equivalence [Har66, Proposition V.2.6 a)],

$$D \colon \mathrm{Perf}(Y_2) \xrightarrow{\sim} D^b_{\mathrm{fid}}(Y_2)^{\mathrm{op}} \tag{8.2}$$

So, we have the following intrinsic characterization of $D^b_{\mathrm{fid}}(Y_1)$ in $D^b_{\mathrm{coh}}(Y_1)$,

$$D^b_{\mathrm{fid}}(Y_1) = \{B \in D^b_{\mathrm{coh}}(Y_1) \mid \mathrm{Hom}(A, B[p]) = 0, \text{for } 0 \ll |p| \text{ and all } A \in D^b_{\mathrm{coh}}(Y_1)\}. \tag{8.3}$$

Indeed, by [Orl06, Proposition 1.11], we have a dual characterization for perfect complexes in $D^b_{\mathrm{coh}}(Y_1)$,

$$\mathrm{Perf}(Y_1) = \{B \in D^b_{\mathrm{coh}}(Y_1) \mid \mathrm{Hom}(B, A[p]) = 0, \text{for } 0 \ll |p| \text{ and all } A \in D^b_{\mathrm{coh}}(Y_1)\},$$

because Y_1 is separated Noetherian of finite Krull dimension and satisfies the resolution property (Orlov calls this (ELF) scheme - note that the proof in loc. cit. does not use that the scheme is defined over a field). This characterization of Perf, together with the equivalences (8.1) and (8.2), implies (8.3).

Recall that Y_1 over S is Gorenstein if, and only if, the dualizing sheaf $f_1^!\mathcal{O}_S$ is invertible (see e.g. [Har66, Exercise V.9.7]). In particular, $f_1^!\mathcal{O}$ is a perfect complex. If now $\Phi\colon D_{\mathrm{qc}}(Y_1) \to D_{\mathrm{qc}}(Y_2)$ is an equivalence, then $\Phi(f_1^!\mathcal{O})$ is a perfect complex in $D_{\mathrm{qc}}(Y_2)$. Furthermore, as Φ restrict to an equivalence on D^b_{coh}, we see by the characterization (8.3) that $\Phi(f_1^!\mathcal{O})$ is contained in $D^b_{\mathrm{fid}}(Y_2)$. Hence, by (8.2), we see that $\Phi(f_1^!\mathcal{O})$ is of the form

$$\Phi(f_1^!\mathcal{O}) \cong D(P) = \mathbf{R}\mathcal{H}om(P, f_2^!\mathcal{O}) \cong P^\vee \otimes^{\mathbf{L}} f_2^!\mathcal{O} \in D^b_{\mathrm{fid}}(Y_2)$$

for some P in $\mathrm{Perf}(Y_2)$. However, $\Phi(f_1^!\mathcal{O})$, and so, $P^\vee \otimes^{\mathbf{L}} f_2^!\mathcal{O}$ is in $\mathrm{Perf}(Y_2)$. Hence, it follows that $f_2^!\mathcal{O}$ is also in $\mathrm{Perf}(Y_2)$. Moreover, since $f_2^!\mathcal{O}^\vee \otimes^{\mathbf{L}} f_2^!\mathcal{O} \cong \mathbf{R}\mathcal{H}om(f_2^!\mathcal{O}, f_2^!\mathcal{O}) \cong \mathcal{O}_{Y_2}$ by the definition of a dualizing complex, it is easy to deduce that $f_2^!\mathcal{O}$ is a shift of a line bundle of Y_2 (see e.g. [Har66, Lemma V.3.3]). Again, by [Har66, Exercise V.9.7], we conclude that Y_2 over S is Gorenstein, which completes the proof. □

9. Elliptic fibrations

We study endofunctors on elliptic fibrations. A technical tool is Proposition 9.4 which shows good behavior of ideal sheaves for diagonals if pulled back along affine morphisms. The main result is Theorem 9.5 which yields an autoequivalence for elliptic fibrations.

Lemma 9.1. *Let $f\colon Y \to X$ be a concentrated proper flat morphism of Noetherian schemes. Denote by C_f the cone of the unit $\mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ where $\Delta_f\colon Y \to Y \times_X Y$ is the diagonal. Set $p, q\colon Y \times_X Y \to Y$ the natural morphisms. If f has affine diagonal, then C_f is relatively perfect for p and q.*

Proof. We prove the case for p because the other is similar (e.g. switch notation). There exists a fibered square

$$\begin{tikzcd} Y \times_X Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & X. \end{tikzcd}$$

Consider the distinguished triangle

$$C_f[-1] \to \mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \to C_f.$$

Since f has affine and proper diagonal, the natural morphism $(\Delta_f)_*\mathcal{O}_Y \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is an isomorphism. Now, if we extend to the long exact sequence in cohomology with the distinguished triangle above, we see that $C_f[-1]$ is a coherent sheaf on $\mathcal{O}_{Y\times_X Y}$. Let $E \in D^b_{\mathrm{qc}}(Y)$. Since q is flat, we know that $\mathbf{L}q^*E \in D^b_{\mathrm{qc}}(Y \times_X Y)$. Consider the distinguished triangle obtain by tensoring with $\mathbf{L}q^*E$,

$$C_f[-1] \otimes^{\mathbf{L}} \mathbf{L}q * E \to \mathcal{O}_{Y\times_X Y} \otimes^{\mathbf{L}} \mathbf{L}q^*E \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}q^*E \to C_f \otimes^{\mathbf{L}} \mathbf{L}q^*E.$$

If we can show that $\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}q^*E \in D^b_{\mathrm{qc}}(Y \times_X Y)$, then we are done. Indeed, since p is proper, flat, and concentrated, we can apply [AJS23, Theorem 2.3]. Now, by projection formula [HR17, Corollary 4.12], we have

$$\begin{aligned}\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \otimes^{\mathbf{L}} \mathbf{L}q^*E &\cong \mathbf{R}(\Delta_f)_*\mathbf{L}\Delta_f^*\mathbf{L}q^*E \\ &\cong \mathbf{R}(\Delta_f)_*\mathbf{L}(q \circ \Delta_f)^*E \\ &\cong \mathbf{R}(\Delta_f)_*E \qquad\qquad (q \circ \Delta_f = 1_Y).\end{aligned}$$

Since Δ_f is affine, it follows that $\mathbf{R}(\Delta_f)_*$ is t-exact and conservative. Hence, $\mathbf{R}(\Delta_f)_*E \in D^b_{\mathrm{qc}}(Y \times_X Y)$, which completes the proof. □

Reminder 9.2. Let X be a Noetherian scheme. An **elliptic X-fibration** is a proper Gorenstein morphism $Y \to X$ from an scheme such that for each $p \in |X|$ one has that $Y \times_X \operatorname{Spec}(\kappa(p))$ is a geometrically integral 1-dimensional k-scheme of genus[3] one with trivial dualizing sheaf.

Example 9.3. Let k be a field. Consider a geometrically integral 1-dimensional k-scheme Y of genus one with trivial dualizing sheaf. Then the natural morphism $Y \times_k Y \to Y$ is an elliptic fibration.

To see, consider a morphism $\operatorname{Spec}(L) \to Y$ from a field. Note that the base change of $Y \times_k Y$ along $\operatorname{Spec}(L) \to Y$ is the base change Y along L/k. As $Y \times_k \operatorname{Spec}(L)$ is geometrically integral (and hence, connected), [Sta26, Tag 0FD2] says that

$$H^0(Y \times_k \operatorname{Spec}(L), \mathcal{O}_{Y \times_k \operatorname{Spec}(L)}) = L.$$

From base change, we see that $Y \times_k \operatorname{Spec}(L)$ is proper and Gorenstein over L. Hence, via, [Sta26, Tag 0A26], $Y \times_k \operatorname{Spec}(L)$ is projective over L. Moreover, [Sta26, Tag 0BY9] tells us that $Y \times_k \operatorname{Spec}(L)$ has genus one. Applying [Sta26, Tag 0FW1], it follows that $Y \times_k \operatorname{Spec}(L)$ has trivial dualizing sheaf. Consequently, the base change of $Y \times_k Y$ over $\operatorname{Spec}(L) \to Y$ is projective, geometrically integral, of Krull dimension one, genus one, and with trivial dualizing sheaf.

More generally, if $Y \to S$ is an elliptic fibration, then $Y \times_S Y \to Y$ is an elliptic fibration. This can be argued in a similar fashion to the case $S = \operatorname{Spec}(k)$ for a field k above. In fact, if $Y \to S$ and $X \to S$ are elliptic fibrations, then the natural projections of $X \times_S Y$ to X and Y are elliptic fibrations. Again, the argument follows analogously to that above.

Proposition 9.4. *Let X be a Noetherian scheme. Let $T \to X$ be an affine morphism from a Noetherian scheme and $f\colon Y \to X$ a flat morphism from a Noetherian scheme. Denote by $\mathcal{I}_{\Delta_f}$ (resp. $\mathcal{I}_{\Delta'_f}$) the ideal sheaf of the diagonal morphism $\Delta_f\colon Y \to Y \times_X Y$ (resp. $\Delta_{f'}\colon Y \times_X T \to Y \times_X Y \times_X T$). Then $\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \cong \mathcal{I}_{\Delta'_f}$ where $t'\colon Y \times_X Y \times_X T \to Y \times_X Y$ is the natural morphism.*

---

[3] We follow [Sta26, Tag 0BY7] for the definition of 'genus'.

Proof. Since f is proper, the diagonal Δ_f is a closed immersion. Consider the commutative diagram obtained by base changes,

$$\begin{tikzcd}
Y \times_X Y \times_X T \arrow[rr, "q'"] \arrow[dr, "p'"] \arrow[dd, "t'"'] & & Y \times_X T \arrow[dr, "f'"] \arrow[dd, dashed, "s"' near end] & \\
& Y \times_X T \arrow[rr, "f'" near start] \arrow[dd, "s" near end] & & T \arrow[dd, "t"] \\
Y \times_X Y \arrow[rr, dashed, "q" near start] \arrow[dr, "p"'] & & Y \arrow[dr, "f"] & \\
& Y \arrow[rr, "f"'] & & S.
\end{tikzcd}$$

Note that base change shows that p, q are flat. We obtain a diagram of fibered squares

$$\begin{tikzcd}
Y \times_X T \arrow[r, "\Delta_{f'}"] \arrow[d, "s"'] & Y \times_X Y \times_X T \arrow[r, "p'"] \arrow[d, "t'"'] & Y \times_X T \arrow[r, "f'"] \arrow[d, "s"] & T \arrow[d, "t"] \\
Y \arrow[r, "\Delta_f"'] & Y \times_X Y \arrow[r, "p"'] & Y \arrow[r, "f"'] & S
\end{tikzcd}$$

where the left most square arises from [Sta26, Tag 04YR]. Since t is affine, base change tells us both s and t' are affine, and hence representable by schemes. By [Sta26, Tag 07U8], there is an isomorphism

$$\begin{aligned}
\mathbf{R}(\Delta_{f'})_*\mathbf{L}s^*\mathcal{O}_Y &\cong \mathbf{R}(\Delta_{f'})_* s^*\mathcal{O}_Y \\
&\cong \mathbf{R}(\Delta_{f'})_*\mathcal{O}_{Y\times_X \mathrm{Spec}(k)} \\
&\cong (t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \\
&\cong (t')^*(\Delta_f)_*\mathcal{O}_Y
\end{aligned}$$

where $(t')^*$ is underived.

We want to show that the natural morphism $(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \to \mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is an isomorphism. This amounts to checking that $\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is concentrated in degree zero. Since $\mathbf{R}t'_*$ is t-exact and conservative, it suffices to check that $\mathbf{R}t'_*\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y$ is concentrated in degree zero. Indeed, t-exactness gives

$$\mathcal{H}^i(\mathbf{R}t'_*\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y) \cong t'_*\mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y),$$

whereas conservativeness shows $t'_*\mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y) \cong 0$ implies $\mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y)$. By projection formula [HR17, Corollary 4.12], we have

$$\mathbf{R}t'_*\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \cong \mathbf{R}t'_*\mathcal{O}_{Y\times_X Y\times_X T} \otimes^{\mathbf{L}} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y.$$

Note that

$$\mathcal{O}_{Y\times_X Y\times_X T} \cong \mathbf{L}(f'\circ p')^*\mathcal{O}_T.$$

By flat base change [HR17, Corollary 4.13], we have

$$\begin{aligned}
\mathbf{R}t'_*\mathcal{O}_{Y\times_X Y\times_X T} &\cong \mathbf{R}t'_*\mathbf{L}(f'\circ p')^*\mathcal{O}_T \\
&\cong \mathbf{L}(f\circ p)^*\mathbf{R}t_*\mathcal{O}_T.
\end{aligned}$$

From projection formula, [HR17, Corollary 4.12], we have

$$\begin{aligned}
&\mathbf{R}t'_*\mathcal{O}_{Y\times_X Y\times_X T} \otimes^{\mathbf{L}} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \\
&\quad\cong \mathbf{L}(f\circ p)^*\mathbf{R}t_*\mathcal{O}_T \otimes^{\mathbf{L}} \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \\
&\quad\cong \mathbf{R}(\Delta_f)_*\mathbf{L}\Delta_f^*\mathbf{L}(f\circ p)^*\mathbf{R}t_*\mathcal{O}_T \\
&\quad\cong \mathbf{R}(\Delta_f)_*\mathbf{L}(f\circ p\circ\Delta_f)^*\mathbf{R}t_*\mathcal{O}_T \\
&\quad\cong \mathbf{R}(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T \qquad\qquad (p\circ\Delta_f = 1_Y).
\end{aligned}$$

Since t is affine, the natural morphism $t_*\mathcal{O}_T \to \mathbf{R}t_*\mathcal{O}_T$ is an isomorphism. Moreover, f is flat, and so the natural morphism $f^*\mathbf{R}t_*\mathcal{O}_T \to \mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is an isomorphism. Hence, $\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is a concentrated in degree zero. Since t' is affine, the natural morphism $(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T \to \mathbf{R}(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is an isomorphism. Thus, $\mathbf{R}(\Delta_f)_*\mathbf{L}f^*\mathbf{R}t_*\mathcal{O}_T$ is concentrated in degree zero.

Consider the distinguished triangle

$$\mathcal{I}_{\Delta_f} \to \mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y \to \mathcal{I}_{\Delta_f}[1].$$

Applying $\mathbf{L}(t')^*$ yields a distinguished triangle

$$\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \to \mathbf{L}(t')^*\mathcal{O}_{Y\times_X Y} \to \mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \to \mathbf{L}(t')^*\mathcal{I}_{\Delta_f}[1].$$

This gives a long exact sequence in cohomology

$$\cdots \to \mathcal{H}^i(\mathbf{L}(t')^*\mathcal{I}_{\Delta_f}) \to \mathcal{H}^i(\mathbf{L}(t')^*\mathcal{O}_{Y\times_X Y}) \to \mathcal{H}^i(\mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y) \to \cdots.$$

Using the identification $\mathbf{L}(t')^*\mathcal{O}_{Y\times_X Y} \cong \mathcal{O}_{Y\times_X Y\times_X T}$ and the previous paragraph, we see that $\mathbf{L}(t')^*\mathcal{I}_{\Delta_f}$ is concentrated in degree zero. There is a morphism of distinguished triangles

$$\begin{tikzcd}
\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \arrow[r] \arrow[d] & \mathbf{L}(t')^*\mathcal{O}_{Y\times_X Y} \arrow[r] \arrow[d] & \mathbf{L}(t')^*\mathbf{R}(\Delta_f)_*\mathcal{O}_Y \arrow[r] \arrow[d] & \mathbf{L}(t')^*\mathcal{I}_{\Delta_f}[1] \arrow[d] \\
\mathcal{I}_{\Delta'_f} \arrow[r] & \mathcal{O}_{Y\times_X Y\times_X T} \arrow[r] & \mathbf{R}(\Delta'_f)_*\mathcal{O}_{Y\times_X T} \arrow[r] & \mathcal{I}_{\Delta'_f}[1].
\end{tikzcd}$$

Since we have shown that the two inner morphisms are isomorphisms, it follows that the morphism $\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \to \mathcal{I}_{\Delta'_f}$ is an isomorphism. □

Theorem 9.5 (cf. [HCS09, Proposition 2.16]). *Let X be a Noetherian scheme. Consider an elliptic X-fibration $f\colon Y \to X$. Then $\Phi_{\mathcal{I}_{\Delta_f}}$ induces an autoequivalence of $D^b_{\mathrm{coh}}(Y)$ where $\mathcal{I}_{\Delta_f}$ is the ideal sheaf of the diagonal $\Delta_f\colon Y \to Y\times_X Y$.*

Proof. We start with an observation. In this paragraph, fix $p \in |X|$. Note that $Y\times_X \operatorname{Spec}(\kappa(p))$ is geometrically connected, of Krull dimension one, genus one, and with trivial dualizing sheaf. Let $\overline{\kappa(p)}$ be an algebraic closure of $\kappa(p)$. Since $Y\times_X \operatorname{Spec}(\kappa(p))$ is geometrically integral, [Sta26, Tag 0FD2] shows that

$$H^0(Y\times_X \operatorname{Spec}(\overline{\kappa(p)}), \mathcal{O}_{Y\times_X \operatorname{Spec}(\overline{\kappa(p)})}) = \overline{\kappa(p)}.$$

By base change, $Y\times_X \operatorname{Spec}(\overline{\kappa(p)})$ is proper and Gorenstein over $\overline{\kappa(p)}$, and so [Sta26, Tag 0A26] implies $Y\times_X \operatorname{Spec}(\overline{\kappa(p)})$ is also projective over $\overline{\kappa(p)}$. Moreover, from [Sta26, Tag 0BY9], $Y\times_X \operatorname{Spec}(\overline{\kappa(p)})$ has genus one. Using [Sta26, Tag 0FW1], we see that $Y\times_X \operatorname{Spec}(\overline{\kappa(p)})$ has trivial dualizing sheaf.

Now, we return to the proof. Since f is proper, the diagonal Δ_f is a closed immersion. Hence, $\mathcal{I}_{\Delta_f}$ is the negative shift of the cone of the unit $\mathcal{O}_{Y\times_X Y} \to \mathbf{R}(\Delta_f)_*\mathcal{O}_Y$. Note that Lemma 9.1 shows that $\mathcal{I}_f$ is relatively perfect over Y. Now, using Theorem 7.15, it suffices to check that the integral transform with kernel being the derived pullback of $\mathcal{I}_{\Delta_f}$ along $\operatorname{Spec}(\overline{\kappa(p)}) \to X$ the algebraic closure of any residue field of a closed point $p \in |X|$. So, we fix such a morphism $t\colon \operatorname{Spec}(k) \to X$ associated to any closed point $p \in |X|$. Note that t is affine [Sta26, Tag 01SI].

By Proposition 9.4, we have $\mathbf{L}(t')^*\mathcal{I}_{\Delta_f} \cong \mathcal{I}_{\Delta'_f}$. Consequently, we have that $\Phi_{\mathbf{L}(t')^*\mathcal{I}_{\Delta_f}}$ is naturally isomorphic to $\Phi_{\mathcal{I}_{\Delta'_f}}$ as endofunctors on $D^b_{\mathrm{coh}}(Y \times_X \operatorname{Spec}(k))$. Since geometrically integral fibers implies irreducible components have the same Krull dimension (i.e. equidimensional fibers), [HCS09, Proposition 2.16 & §3.4.1] implies that $\Phi_{\mathcal{I}_{\Delta'_f}}$ is an autoequivalence of $D^b_{\mathrm{coh}}(Y \times_X \operatorname{Spec}(k))$. Thus, we complete the proof. □

Remark 9.6. Theorem 9.5 does not require an separatedness.

10. Singularity category

We study variants of our earlier results for the singularity category of an scheme. In particular, we establish the openness of the locus of points on the base scheme for which an integral transform induces an exact functor on singularity categories. Furthermore, we show that there is no fiberwise criterion for determining when an integral transform induces a fully faithful functor or an equivalence on singularity categories.

Lemma 10.1. *Let X be a Noetherian scheme. For any $E \in D^b_{\mathrm{coh}}(X)$, the collection of $p \in |\mathcal{X}|$ such that $\mathbf{L}s^*E$ is perfect, where $s\colon \operatorname{Spec}(\kappa(p)) \to X$, is open in $|X|$.*

Proof. The problem is affine local. In such a case, apply [Let21, Proposition 2.5] and [AJS23, Theorem 2.3]. □

Proposition 10.2. *Consider Setup 6.1. Let $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$. Then the collection of $p \in |S|$ for which $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(\kappa(p))$ (resp. $Y_2 \times_S \operatorname{Spec}(\kappa(p))$) forms an open substack of S.*

Proof. Denote by V the collection of $p \in |S|$ such that $\mathbf{L}(t')^*K$ is relatively perfect over $Y_1 \times_S \operatorname{Spec}(\kappa(p))$. Recall that each $D_{\mathrm{qc}}(Y_i)$ is singly compactly generated by some G_i. We prove the case of being relatively perfect over $Y_1 \times_S \operatorname{Spec}(\kappa(p))$ because the other is symmetric.

Appealing to Lemma 10.1, we know that the collection U of $q \in |Y_1|$ such that

$$\mathbf{L}s^*\mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^*G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^*G_2)$$

is perfect, where $s\colon \operatorname{Spec}(\kappa(q)) \to Y_1$ represents q, is open in $|Y_1|$. Since f_1 is proper, it follows that $f(|Y_1| \setminus U)$ is closed in $|S|$. We claim that $f(|Y_1| \setminus U) = |S| \setminus V$. In doing so, we show that V is open.

Let $p \in |S| \setminus V$. Consider the natural morphism $t\colon \operatorname{Spec}(\kappa(p)) \to S$ for p. Consider the fibered square

$$\begin{tikzcd} Y_1 \times_S \operatorname{Spec}(\kappa(p)) \arrow[r, "g_1"] \arrow[d, "t_1"'] & \operatorname{Spec}(\kappa(p)) \arrow[d, "t"] \\ Y_1 \arrow[r, "f_1"'] & S. \end{tikzcd}$$

Since $p \notin V$, it follows that $\mathbf{L}(t')^* K$ is not relatively perfect over $Y_1 \times_S \operatorname{Spec}(\kappa(p))$. This implies that

$$\mathbf{L}t_1^* \mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^* G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* G_2) \notin \operatorname{Perf}(Y_1 \times_S \operatorname{Spec}(\kappa(p))).$$

Using [AJS23, Theorem 2.3], we can find $q \in |Y_1 \times_S \operatorname{Spec}(\kappa(p))|$ such that

$$\mathbf{L}(t_1 \circ s)^* \mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^* G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* G_2) \notin D^b_{\operatorname{coh}}(\kappa(q))$$

where $s\colon \operatorname{Spec}(\kappa(q)) \to Y_1 \times_S \operatorname{Spec}(\kappa(q))$ is the natural morphism. Now, $(t_1 \circ s)(q) = p$ and $t_1(q) \in |Y_1| \setminus U$, which implies $p \in f_1(|Y_1| \setminus U)$. This shows that $|S| \setminus V \subseteq f(|Y_1| \setminus U)$.

Conversely, let $p \in f(|Y_1| \setminus U)$. Choose $q \in |Y_1| \setminus U$ such that $f_1(q) = p$. Suppose $t\colon \operatorname{Spec}(\kappa(p)) \to S$ is the natural morphism. Consider the fibered square

$$\begin{tikzcd} Y_1 \times_S \operatorname{Spec}(\kappa(p)) \arrow[r, "g_1"] \arrow[d, "t_1"'] & \operatorname{Spec}(\kappa(p)) \arrow[d, "t"] \\ Y_1 \arrow[r, "f_1"'] & S. \end{tikzcd}$$

By [Sta26, Tag 04XH], there is a $q' \in |Y_1 \times_S \operatorname{Spec}(\kappa(p))|$ such that $t_1(q') = q$. Indeed, the morphism of the fiber product in category of sets says $(q, \{pt\}) \in |Y_1| \times_{|S|} |\operatorname{Spec}(\kappa(p))|$ is a lift of q along the natural morphism. Yet, [Sta26, Tag 04XH] gives us a lift of $(q, \{pt\})$ and clearly t_1 factors through the natural morphism on sets $|Y_1| \times_{|S|} |\operatorname{Spec}(\kappa(p))| \to |Y_1|$. Set $s\colon \operatorname{Spec}(\kappa(q')) \to Y_1 \times_S \operatorname{Spec}(\kappa(p))$ to the natural morphism. Since $q \notin U$, it follows that

$$\mathbf{L}(t_1 \circ s)^* \mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^* G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* G_2) \notin D^b_{\operatorname{coh}}(\kappa(q')).$$

Consequently, we obtain

$$\mathbf{L}t_1^* \mathbf{R}(f_2')_*(K \otimes^{\mathbf{L}} \mathbf{L}(f_2')^* G_1 \otimes^{\mathbf{L}} \mathbf{L}(f_1')^* G_2) \notin \operatorname{Perf}(Y_1 \times_S \operatorname{Spec}(\kappa(p))).$$

Indeed, use [AJS23, Theorem 2.3]. Thus, $p \notin V$, and hence $f(|Y_1| \setminus U) \subseteq |S| \setminus V$. □

Lemma 10.3. *Consider Setup 6.1.. Let $K \in D^b_{\operatorname{coh}}(Y_1 \times_S Y_2)$ be relatively perfect over each Y_i. Assume K' in Corollary 6.13 is relatively perfect over each Y_i. Denote by η the unit of the adjoint pair Φ_K and its right adjoint $\Phi_{K'}$ on D^b_{coh}. Then the induced exact functor $\widetilde{\Phi}_K$ on D_{sg} is fully faithful if, and only if,* $\operatorname{cone}(\eta_E) \in \operatorname{Perf}(Y_1)$ *for all $E \in D^b_{\operatorname{coh}}(Y_1)$.*

Proof. Since K and K' are relatively perfect over each Y_i, the adjoint pair Φ_K and its right adjoint $\Phi_{K'}$ on D^b_{coh} restricts to D_{sg}. In particular, the unit of the restricted adjoint pair is the image of the unit in D^b_{coh}. Thus, $\widetilde{\Phi}_K$ is fully faithful if, and only if, the cone the unit of the restricted adjoint pair is the zero object in $D_{\operatorname{sg}}(Y_1)$. However, the latter condition is equivalent to $\operatorname{cone}(\eta_E) \in \operatorname{Perf}(Y_1)$ for all $E \in D^b_{\operatorname{coh}}(Y_1)$. □

Lemma 10.4. *Consider Setup 6.1. Let* $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ *be relatively perfect over each* Y_i*. Assume* K' *in Corollary 6.13 is relatively perfect over each* Y_i*. Denote by* ϵ *the counit of the adjoint pair* Φ_K *and its right adjoint* $\Phi_{K'}$ *on* D^b_{coh}*. Then the induced exact functor* $\widetilde{\Phi}_{K'}$ *on* D_{sg} *is fully faithful if, and only if,* $\operatorname{cone}(\epsilon_E) \in \operatorname{Perf}(Y_2)$ *for all* $E \in D^b_{\mathrm{coh}}(Y_2)$*.*

Proof. This is dual to Lemma 10.3. □

Proposition 10.5. *Consider Setup 6.1 where* $S = S$ *is a regular scheme and* t *is an affine faithfully flat morphism. Let* $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ *be relatively perfect over each* Y_i*. Assume* K' *in Corollary 6.13 is relatively perfect over each* Y_i*. If* $\Phi_{\mathbf{L}(t')^*K}$ *induces a fully faithful functor (resp. an equivalence) on* D_{sg}*, then* Φ_K *induces a fully faithful (resp. an equivalence) on* D_{sg}*.*

Proof. We prove the claim for fully faithfulness because the other is analagous. Denote by η the unit of the adjoint pair Φ_K and its right adjoint $\Phi_{K'}$ on D^b_{coh}. By Lemma 10.4, it suffices to show $\operatorname{cone}(\eta_E) \in \operatorname{Perf}(Y_1)$ for all $E \in D^b_{\mathrm{coh}}(Y_1)$. By base change, t_1 is an affine faithfully flat morphism. By (7.10), it follows that

$$\mathbf{L}t_1^* \operatorname{cone}(\eta_E) \cong \operatorname{cone}(\eta'_{\mathbf{L}t_1^*E})$$

where η' is the unit of $\Phi_{\mathbf{L}t_1K}$ and its right adjoint $\Phi_{\mathbf{L}t_1K'}$ on D^b_{coh}. Choose a smooth surjection morphism $s\colon U \to Y_1 \times_S \mathcal{T}$ by a scheme. Then $\mathbf{L}(s \circ t_1)^* \operatorname{cone}(\eta_E) \in \operatorname{Perf}(U)$, and so, [DLMP25, Lemma 2.3] implies $\operatorname{cone}(\eta_E) \in \operatorname{Perf}(Y_1)$ as desired. □

Lemma 10.6. *Let* $f\colon Y \to X$ *be a morphism of quasi-compact quasi-separated schemes. For any* $E \in D^b_{\mathrm{qc}}(X)$*, the following are equivalent:*

(1) $E \otimes^{\mathbf{L}} \mathbf{L}f^* D^b_{\mathrm{qc}}(X) \subseteq D^b_{\mathrm{qc}}(X)$

(2) *there exists* $[a,b] \subseteq \mathbb{Z}$ *such that* $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \cong 0$ *for all* $j \notin [a,b]$ *and* $M \in \operatorname{Qcoh}(X)$*.*

Proof. This is argued like Lemma 3.2 but we add it anyways. First, we show (1) $\Longrightarrow$ (2). Assume the contrary. Then for each $n \geq 1$ there exists an $M_n \in \operatorname{Qcoh}(Y)$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \ncong 0$ for some $j \notin [-n,n]$. Set $M := \oplus_{n\geq 1} M_n$. By hypothesis, $E \otimes^{\mathbf{L}} \mathbf{L}f^*M \in D^b_{\mathrm{qc}}(Y)$, and hence there exists $[a,b] \subseteq \mathbb{Z}$ such that $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M) \cong 0$ for all $j \notin [a,b]$. After shifting, if necessary, we can impose $a = 0$. Since $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M_n)$ is a direct summand of $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M)$ for each $n \geq 1$, $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M_n) \cong 0$ for all $j \notin [0,b]$ and $n \geq 1$. This leads to a contradiction. Indeed, for $n > b$ there exists $j \notin [0,n]$ with $\mathcal{H}^j(E \otimes^{\mathbf{L}} \mathbf{L}f^*M_n) \ncong 0$.

Next, we prove that (2) $\Longrightarrow$ (1). Let $B \in D^b_{\mathrm{qc}}(X)$. By Lemma 3.1, B is finitely built by its cohomology sheaves $\mathcal{H}^j(B)$. Moreover, the hypothesis implies $\mathbf{L}f^*\{\mathcal{H}^i(B)\}_{i\in\mathbb{Z}} \subseteq D^b_{\mathrm{qc}}(Y)$. Since $B \in \langle \{\mathcal{H}^i(B)\} \rangle$, it follows that

$$E \otimes^{\mathbf{L}} \mathbf{L}f^*B \in \langle \{E \otimes^{\mathbf{L}} \mathbf{L}f^*\mathcal{H}^i(B)\} \rangle \subseteq D^b_{\mathrm{qc}}(Y).$$

This completes the proof. □

Reminder 10.7. A morphism $f\colon Y \to X$ of schemes has **finite tor-dimension** if there exists an interval $[a,b]$ such that $\mathcal{H}^i(\mathbf{L}f^*E) \cong 0$ for all $E \in \operatorname{Qcoh}(X)$ and $i \notin \mathbb{Z} \cap [a,b]$.

Lemma 10.8. *Let* X *be a quasi-compact quasi-separated scheme. For any* $E \in D_{\mathrm{qc}}(X)$*, the following are equivalent:*

(1) $E \cong 0$
(2) $\mathbf{L}f^*E \cong 0$ *for all flat morphisms* $f\colon Y \to X$ *of schemes*
(3) there exists a smooth surjective morphism $f\colon Y \to X$ *of schemes such that* $\mathbf{L}f^*E \cong 0$
(4) there exists a flat surjective morphism $f\colon Y \to X$ *of schemes such that* $\mathbf{L}f^*E \cong 0$.

Proof. It is straightforward to check that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). Moreover, by Lemma 7.9, (4) $\implies$ (1). ☐

Lemma 10.9. *Let* $f\colon Y \to X$ *be a finite morphism of finite tor-dimension between Noetherian schemes. Consider a concentrated flat morphism* $g\colon U \to X$ *from a Noetherian scheme. Denote by* $p\colon U \times_X Y \to U$ *and* $q\colon U \times_X Y \to Y$ *the natural morphisms. Then* p *is finite and of finite tor-dimension.*

Proof. By base change, p is finite. We show that it is of finite tor-dimension. Flat base change (see [HR17, Corollary 4.13]) implies

$$\mathbf{R}p_*\mathcal{O}_{U\times_X Y} \cong \mathbf{R}p_*\mathbf{L}q^*\mathcal{O}_Y \cong \mathbf{L}g^*\mathbf{R}f_*\mathcal{O}_Y \in \operatorname{Perf}(\mathcal{O}_U).$$

Choose a smooth surjection $s\colon V \to U$. Consider the fibered square

$$\begin{tikzcd} V \times_X Y \arrow[r, "s'"] \arrow[d, "p'"'] & U \times_X Y \arrow[d, "p"] \\ V \arrow[r, "s"'] & U. \end{tikzcd}$$

Here, $V \times_X Y$ is a Noetherian scheme. Applying flat base change again, we obtain $\mathbf{R}p'_*\mathcal{O}_{U\times_X Y} \in \operatorname{Perf}(U)$. Since p' is finite by base change, [LH09, Example 4.7.3(d)] implies p' has finite tor-dimension. Hence, there is an interval $[a,b]$ such that $\mathscr{H}^i(\mathbf{L}(p')^*E) \cong 0$ for all $i \notin [a,b] \cap \mathbb{Z}$ and $E \in \operatorname{Qcoh}(U)$. Let $E \in \operatorname{Qcoh}(U)$. Then $\mathscr{H}^i(\mathbf{L}(p \circ s')^*E) \cong 0$ for all $i \notin [a,b] \cap \mathbb{Z}$. Since s' is faithfully flat, it is t-exact with respect to the standard t-structures. Hence, by Lemma 10.8, it follows that $\mathscr{H}^i(\mathbf{L}p^*E) \cong$ for all $i \notin [a,b] \cap \mathbb{Z}$. Therefore, Lemma 10.6 implies p has finite tor-dimension. ☐

Proposition 10.10. *Let* $\mathbb{S}_{\mathbf{e}}$ *be the* 2*-category of schemes which are separated and of finite presentation over a fixed algebraically closed field* k*. Consider Setup 6.1 in* $\mathbb{S}_{\mathbf{e}}$ *where* $S = S$ *is a regular scheme. Let* $K \in D^b_{\operatorname{coh}}(Y_1 \times_S Y_2)$ *be relatively perfect over each* Y_i*. Assume* K' *in Corollary 6.13 is relatively perfect over each* Y_i*. If for every* $t\colon \operatorname{Spec}(k) \to S$ *from a field that maps to a closed point* $p \in |S|$ *one has that* $\Phi_{\mathbf{L}(t')^*K}$ *induces a fully faithful (resp. an equivalence) functor on* D_{sg}*, then* Φ_K *induces a fully faithful (resp. an equivalence) functor on* D_{sg}*.*

Proof. We prove the claim in the fully faithful case because the equivalence case is analogous. Denote by η the unit of the adjunction between Φ_K and its right adjoint $\Phi_{K'}$ on D^b_{coh}. By Lemma 10.4, it suffices to show $\operatorname{cone}(\eta_E) \in \operatorname{Perf}(Y_1)$ for all $E \in D^b_{\operatorname{coh}}(Y_1)$.

Let $p \in |Y_1|$ be a closed point. Consider the natural morphism $s\colon \operatorname{Spec}(k') \to Y_1$ where $\kappa(p) =: k'$. Since f_1 induces a closed morphism $|Y_1| \to |S|$, we know that $f_1(p) \in |S|$ is closed. Hence, there exist a field ℓ and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(\ell) \arrow[r, "u"] \arrow[d, "g"'] & \operatorname{Spec}(\kappa(f_1(p))) \arrow[d, "t"] \\ \operatorname{Spec}(k') \arrow[r, "f_1 \circ s"'] & S. \end{tikzcd}$$

Since k is algebraically closed, $\kappa(f_1(p)) \cong k \cong k'$, and so we can choose g and u to be isomorphisms. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(\ell) \arrow[dr, "h"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "g\circ s"'] & & \\
& Y_1 \times_S \operatorname{Spec}(\kappa(f_1(p))) \arrow[r, "f_1'"] \arrow[d, "t_1"] & \operatorname{Spec}(\kappa(f_1(p))) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"'] & S.
\end{tikzcd}$$

Note that u and g are of finite type. Since S is regular and t is a closed immersion, it follows that t is of finite tor-dimension. By Lemma 10.9, it follows that t_1 is of finite tor-dimension. Hence, $\mathbf{L}t_1^*$ preserves complexes with bounded and coherent cohomology. By (7.10), it follows that

$$\mathbf{L}t_1^* \operatorname{cone}(\eta_E) \cong \operatorname{cone}(\eta'_{\mathbf{L}t_1^*E})$$

where η' is the unit of $\Phi_{\mathbf{L}t_1^*K}$ and its right adjoint $\Phi_{\mathbf{L}t_1^*K'}$ on D^b_{coh}. By hypothesis, $\Phi_{\mathbf{L}t_1K}$ is fully faithful. Hence, by Lemma 10.4, $\mathbf{L}t_1^* \operatorname{cone}(\eta_E)$ is perfect, and so, $\mathbf{L}h^*\mathbf{L}t_1^* \operatorname{cone}(\eta_E)$ is perfect. Consequently, by Lemma 10.1, we have shown that p belongs to the collection of $q \in |Y_1|$ such that $\mathbf{L}r^* \operatorname{cone}(\eta_E)$ is perfect, where $r\colon \operatorname{Spec}(L) \to Y_1$ is a representative of q. Since this holds for all closed points, it follows from [AJS23, Theorem 2.3] that $\operatorname{cone}(\eta_E)$ is perfect, which completes the proof. □

Example 10.11. The following example shows that the condition 'for every representative t' in Proposition 10.10 cannot be weakened to 'there exists a representative t'. Moreover, it exhibits that the base field being algebraically closed is important. Indeed, suppose that this replacement were valid. Let $k := \mathbb{F}_3(t)$ where t is transcendental. Consider the projective plane curve C over k given by the equation $y^2z + x^3 - tz^3 = 0$. This is a regular Noetherian scheme as it is normal and of Krull dimension one. However, after base change of C along the field extension $\ell := \mathbb{F}_3(t^{\frac{1}{3}})$, $C \times_k \operatorname{Spec}(\ell)$ is a singular projective curve [Kol11, Remark 16]. Denote by t' the natural base change morphism

$$C \times_k C \times_k \operatorname{Spec}(\ell) \to C \times_k C.$$

Since C is regular, Φ_P induces an autoequivalence on $D_{\mathrm{sg}}(C)$ for every perfect complex P on $C \times_k C$. By our assumption, it follows that $\Phi_{\mathbf{L}(t')^*P}$ induces an autoequivalence on $D_{\mathrm{sg}}(C \times_k \operatorname{Spec}(\ell))$. If we set P to be the zero object, it follows that $D_{\mathrm{sg}}(C \times_k \operatorname{Spec}(\ell))$ consists only of zero objects. Hence, $C \times_k \operatorname{Spec}(\ell)$ must be regular, which is absurd.

Example 10.12. The following example shows that the regularity assumption on the base scheme in Proposition 10.10 is essential. Assume the statement were true for some Noetherian scheme S that is not regular. Consider the projective N-space $f\colon \mathbb{P}^N_S \to S$ over S. Here f is smooth and proper. Consider the integral transform Φ on $\mathbb{P}^N_S$ with kernel given by a zero object K. Its right adjoint is again the integral transform with kernel equal to a zero object. In particular, these functors form an adjoint pair on D_{sg}. Let $t\colon \operatorname{Spec}(k) \to S$ be a representative of finite type for a closed point of $|S|$. The base change of Φ_K along t induces an autoequivalence on $D_{\mathrm{sg}}(\mathbb{P}^N_k)$. Indeed, $\mathbb{P}^N_k$ is regular, and hence, $D_{\mathrm{sg}}(\mathbb{P}^N_k)$ consists only of zero objects. By the assumption, this induces an autoequivalence on $D_{\mathrm{sg}}(\mathbb{P}^N_S)$, which implies every object of $D_{\mathrm{sg}}(\mathbb{P}^N_S)$ is zero. Thus, $\mathbb{P}^N_S$ is regular, and hence S is regular, which is absurd.

11. Openness of locii

We show that the locus of points where equivalences or fully faithfuless occurs for fibers forms an open subset on the base. This generalizes [HP24, Corollary 5.9] in two ways: first, we do not require perfect kernels, and two the arguments differ since that of [AL12] are not available.

Theorem 11.1. *Let $f_i \colon Y_i \to S$ be proper morphisms of Noetherian schemes. If $K \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ is relatively perfect over each Y_i, then the collection of $p \in S$, for which the base change of Φ_K along $\operatorname{Spec}(\kappa(p)) \to S$ is an equivalence (resp. fully faithful) on D_{qc}, is an open subscheme.*

Proof. By Corollary 6.13, there exists $K' \in D^b_{\mathrm{coh}}(Y_1 \times_S Y_2)$ such that Φ_K is right adjoint to $\Phi_{K'}$. Denote by η the unit for this adjunction. We prove that η is a natural isomorphism. In turn, we obtain that Φ_K is fully faithful. Let G be a compact generator for $D_{\mathrm{qc}}(Y_1)$. Define $\mathcal{T}$ to be the category of objects $E \in D_{\mathrm{qc}}(Y_1)$ satisfying η_E is an isomorphism. It it straightforward to check that $\mathcal{T}$ is localizing. If we can show that η_G is an isomorphism, then we are done because $G \in \mathcal{T}$.

Define L to be the collection of $p \in S$ for which the base change of Φ_K along the natural morphism $\operatorname{Spec}(\kappa(p)) \to S$ is fully faithful (or an equivalence) on D_{qc}. We claim that

$$S \setminus L = f_1(\operatorname{supp}(\operatorname{cone}(\eta_G))).$$

Since f_1 is a closed morphism, this would show that the complement of L is closed, which would complete the proof.

Let $p \in f_1(\operatorname{supp}(\operatorname{cone}(\eta_G)))$. Choose $p' \in \operatorname{supp}(\operatorname{cone}(\eta_G))$ such that $f_1(p') = p$. Consider the fibered square

$$\begin{tikzcd}
\operatorname{Spec}(\kappa(p')) \arrow[dr, "r"] \arrow[drr, bend left, "g"] \arrow[ddr, "h"'] & & \\
& Y_1 \times_S \operatorname{Spec}(\kappa(p)) \arrow[r, "f_1'"] \arrow[d, "t_1"'] & \operatorname{Spec}(\kappa(p)) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"'] & S
\end{tikzcd}$$

where t is the natural morphism. Since $p' \in \operatorname{supp}(\operatorname{cone}(\eta_G))$, we have $\mathbf{L}h^* \operatorname{cone}(\eta_G)) \neq 0$. Hence, $\mathbf{L}t_1^* \operatorname{cone}(\eta_G) \neq 0$, and so by (7.13),

$$0 \ncong \mathbf{L}t_1^* \operatorname{cone}(\eta_G) \cong \operatorname{cone}(\mathbf{L}t_1^* \eta_G) \cong \operatorname{cone}(\eta'_{\mathbf{L}t_1^* G})$$

where η' is the unit for the adjoint pair $\Phi_{\mathbf{L}(t')^* K}$ and $\Phi_{\mathbf{L}(t')^* K'}$. Here $t' \colon Y_1 \times_S Y_2 \times_S \operatorname{Spec}(\kappa(p)) \to Y_1 \times_S Y_2$ is the natural morphism. Now, this cone above is bounded and pseudocoherent, which implies that $\Phi_{\mathbf{L}(p')^* K}$ is not fully faithful on D_{qc}. Indeed, this shows that the unit is not an isomorphism for a compact generator. Consequently, $p \notin L$, and so $f_1(\operatorname{supp}(\operatorname{cone}(\eta_G))) \subseteq S \setminus L$.

We check the reverse inclusion. Let $p \in S \setminus L$. Consider the fibered square

$$\begin{tikzcd}
\operatorname{Spec}(\kappa(p')) \arrow[drr, bend left, "g"] \arrow[dr, "r"] \arrow[ddr, "h"'] & & \\
& Y_1 \times_S \operatorname{Spec}(\kappa(p)) \arrow[r, "f_1'"] \arrow[d, "t_1"'] & \operatorname{Spec}(\kappa(p)) \arrow[d, "t"] \\
& Y_1 \arrow[r, "f_1"'] & S
\end{tikzcd}$$

where t is the natural morphism. Set η' to be the unit for the adjoint pair $\Phi_{\mathbf{L}(t')^*K}$ and $\Phi_{\mathbf{L}(t')^*K'}$. Here $t'\colon Y_1 \times_S Y_2 \times_S \operatorname{Spec}(\kappa(p)) \to Y_1 \times_S Y_2$ is the natural morphism. Our choice of p implies that $\Phi_{\mathbf{L}(t')^*K}$ is not an equivalence (resp. fully faithful) on D_{qc}. Since t is affine [Sta26, Tag 01SI], then so is t_1, and so $\mathbf{L}t_1^*G$ is a compact generator for $D_{\mathrm{qc}}(Y_1 \times_S \operatorname{Spec}(\kappa(p)))$. Then $\eta'_{\mathbf{L}t_1^*G}$ is not an isomorphism, and so, $\operatorname{cone}(\eta'_{\mathbf{L}t_1^*G})$ is nonzero. However, this cone is in D^b_{coh} because K is relatively perfect over each Y_i (see Corollary 6.13), and so there exists some $q \in \operatorname{supp}(\operatorname{cone}(\eta'_{\mathbf{L}t_1^*G}))$ for which $\mathbf{L}h^* \operatorname{cone}(\eta'_{\mathbf{L}t_1^*G})$ is nonzero where $h\colon \operatorname{Spec}(\kappa(q)) \to Y_1 \times_S \operatorname{Spec}(\kappa(p))$ the natural morphism. Now, we have a commutative square

$$\begin{tikzcd}
\operatorname{Spec}(\kappa(q)) \arrow[r, "h"'] \arrow[d, "g"] & Y_1 \times_S \operatorname{Spec}(\kappa(p)) \arrow[d, "T_1"'] \\
\operatorname{Spec}(\kappa(t_1(q))) \arrow[r, "h'"] & Y_1
\end{tikzcd}$$

where h' is the natural morphism. By (7.13), it follows that

$$0 \not\cong \mathbf{L}(h' \circ g)^* \operatorname{cone}(\eta_G) \cong \mathbf{L}(t_1 \circ h)^* \operatorname{cone}(\eta_G) \cong \operatorname{cone}(\mathbf{L}(t_1 \circ h)^* \eta_G) \cong \mathbf{L}h^* \operatorname{cone}(\eta'_{\mathbf{L}t_1^*G}).$$

Since g is faithfully flat, we have that

$$0 \not\cong \mathbf{L}g^* \operatorname{cone}(\eta_G).$$

In other words, $t_1(q) \in \operatorname{supp}(\eta_G)$. Consequently,

$$p = f_1'(t_1(q)) \in f_1(\operatorname{supp}(\operatorname{cone}(\eta_G))).$$

□

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