REVISITING PERFECT COMPLEXES ON ALGEBRAIC STACKS

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Abstract. We establish single compact generation of the derived category of quasi-coherent sheaves for algebraic stacks admitting a closed immersion into a regular Noetherian algebraic stack with quasi-finite and locally separated diagonal.

1. Introduction

Compact generation of triangulated categories associated to an algebraic stack plays a central role in modern algebraic geometry. This includes the derived category D_{qc} of complexes with quasi-coherent cohomology. In several important cases, D_{qc} is singly compactly generated; that is, there exists a single compact object that generates the entire triangulated category under shifts, cones, and small coproducts.

For quasi-compact and quasi-separated schemes, this property was established by Bondal and Van den Bergh [BV03, Theorem 3.1.1]. Subsequently, Hall and Rydh proved compact generation for quasi-compact and quasi-separated algebraic stacks whose diagonal is quasi-finite and separated [HR17, Theorem A]. That work subsumed and extended a number of earlier results for Deligne–Mumford stacks and related classes [Toë12, Kri09, BZFN10]. However, in positive characteristic, compact generation of D_{qc} fails at times [HNR19].

This brings attention to our main result.

Theorem 1.1. *Let $\mathcal{X}$ be a regular Noetherian algebraic stack with quasi-finite and locally separated diagonal. Then $D_{\mathrm{qc}}(\mathcal{Z})$ is singly compactly generated for any closed immersion $\mathcal{Z} \to \mathcal{X}$.*

Proof. This follows from Lemma 3.5 and Proposition 3.6. □

While locally separatedness on the diagonal might be considered slightly milder than other results in the literature, Theorem 1.1 applies to a broader class of examples without separated diagonals, e.g. if $\mathcal{X}$ is a Deligne–Mumford stack. To the best of our knowledge, Theorem 1.1 is new, even for cases in arbitrary characteristic (cf. [HR18, Theorem 7.4]). Moreover, when combined with [HNR19, Theorem 1.2], it implies that for any such algebraic stack in Theorem 1.1, the natural functor $D(\mathrm{Qcoh}(\mathcal{X})) \to D_{\mathrm{qc}}(\mathcal{X})$ is an equivalence.

The proof revisits the approach of [HR17, Theorem A]. There are two main ingredients: first, the finite flat presentations of algebraic stacks with quasi-finite and locally separated diagonal developed in [Ryd11]; and second, a systematic use of recollements of triangulated categories to glue compact generation across the relevant geometric steps. One a related note, [HR17, Remark 8.3] asked whether the so-called 'Thomason condition' ascends along quasi-affine morphisms. While we do not resolve this question in full generality, we show

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that sufficient control of duality allows one to establish ascent in the setting of closed immersions into regular stacks.

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2. Preliminaries

2.1. **Algebraic stacks**. Our conventions for algebraic stacks are those of [Sta25]. For the derived pullback/pushforward adjunction, we adopt the conventions of [HR17, §1] and [Ols07, LO08a, LO08b]. Unless otherwise specified, symbols such as X, Y, etc. denote schemes or algebraic spaces, while $\mathcal{X}$, $\mathcal{Y}$, etc. denote algebraic stacks. In this subsection, let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack.

Categories. We specify the triangulated categories that appear in our work. Let $\operatorname{Mod}(\mathcal{X})$ denote the Grothendieck abelian category of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules on the lisse-étale site of $\mathcal{X}$. Define $\operatorname{Qcoh}(\mathcal{X})$ to be the strictly full subcategory (i.e. full and closed under isomorphisms) of $\operatorname{Mod}(\mathcal{X})$ consisting of quasi-coherent sheaves. Set $D(\mathcal{X}) := D(\operatorname{Mod}(\mathcal{X}))$ for the derived category of $\operatorname{Mod}(\mathcal{X})$. Denote by $D_{\mathrm{qc}}(\mathcal{X})$ the full subcategory of $D(\mathcal{X})$ consisting of complexes with quasi-coherent cohomology sheaves. Finally, let $\operatorname{Perf}(\mathcal{X})$ denote the full subcategory of perfect complexes in $D_{\mathrm{qc}}(\mathcal{X})$.

Support. Let $M \in \operatorname{Qcoh}(\mathcal{X})$. Set $\operatorname{supp}(M) := p\big(\operatorname{supp}(p^* M)\big) \subseteq |\mathcal{X}|$ where $p\colon U \to \mathcal{X}$ is any smooth surjective morphism from a scheme. One checks that this definition is independent of the choice of p. Now, given any $E \in D_{\mathrm{qc}}(\mathcal{X})$, let

$$\operatorname{supp}(E) := \bigcup_{j\in\mathbb{Z}} \operatorname{supp}\left(\mathcal{H}^j(E)\right) \subseteq |\mathcal{X}|.$$

This subset of $|\mathcal{X}|$ is called the **support** of E.

Concentratedness. A morphism of algebraic stacks is called **concentrated** if it is quasi-compact and quasi-separated, and if for every base change along a quasi-compact and quasi-separated morphism, the derived pushforward has finite cohomological dimension. For instance, by [HR17, Lemma 2.5(3)], morphisms which are representable by algebraic spaces are concentrated. An algebraic stack is **concentrated** if it is quasi-compact and quasi-separated, and its structure morphism to $\operatorname{Spec}(\mathbb{Z})$ is concentrated. In fact, a quasi-compact and quasi-separated algebraic stack $\mathcal{X}$ is concentrated if and only if any (hence all) of the following equivalent conditions hold: $\operatorname{Perf}(\mathcal{X}) = D_{\mathrm{qc}}(\mathcal{X})^c$, $\mathcal{O}_{\mathcal{X}} \in D_{\mathrm{qc}}(\mathcal{X})^c$, or $\mathbf{R}\Gamma\colon D_{\mathrm{qc}}(\mathcal{X}) \to D_{\mathrm{qc}}(\operatorname{Spec}(\mathbb{Z}))$ commutes with small coproducts. See [HR17, §2 & Remark 4.6] for details.

Perfect complexes. Perfect complexes may be defined on any ringed site [Sta25, Tag 08G4], in particular on the lisse-étale site of $\mathcal{X}$. A complex is **strictly perfect** if it is a bounded complex whose terms are direct summands of finite free modules; it is **perfect** if it is locally strictly perfect. Let $\operatorname{Perf}(\mathcal{X})$ denote the triangulated subcategory of $D_{\mathrm{qc}}(\mathcal{X})$ consisting of perfect complexes. In general, the compact objects of $D_{\mathrm{qc}}(\mathcal{X})$ are perfect complexes [HR17, Lemma 4.4], although the converse need not hold. The two notions coincide precisely when the algebraic stack $\mathcal{X}$ is concentrated. Any compact object of $D_{\mathrm{qc}}(\mathcal{X})$ is a perfect complex and the support of a perfect complex has quasi-compact complement (see [HR17, Lemmas 4.4 & 4.8]).

Thomason condition. In general, $D_{\mathrm{qc}}(\mathcal{X})$ need not be compactly generated (for instance, this fails for $D_{\mathrm{qc}}(B_k\mathbb{G}_a)$ when k is a field of positive characteristic; see [HNR19, Proposition 3.1]). A related notion is the 'Thomason condition' which was introduced in [HR17]. We say that $\mathcal{X}$ satisfies the β-**Thomason condition**, for some cardinal β, if $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated by a set of cardinality at most β, and if for every closed subset $Z \subseteq |\mathcal{X}|$ with quasi-compact complement there exists a perfect complex $P \in \mathrm{Perf}(\mathcal{X})$ such that $\mathrm{supp}(P) = Z$. For brevity, we say $\mathcal{X}$ is **Thomason** if it satisfies the β-Thomason condition for some cardinal β. For example, any quasi-compact quasi-separated scheme satisfies the Thomason condition (see e.g. [Sta25, Tag 09IS & Tag 08ES]).

2.2. **Recollements**. We briefly recall the notion of a recollement. See [BBDG18, §1.4] for details. A **recollement** is a commutative diagram of triangulated categories and exact functors of the form

$$\begin{tikzcd} \mathcal{T} \arrow[r, "I" description] & \mathcal{K} \arrow[l, bend right, "I_\lambda"'] \arrow[l, bend left, "I_\rho"] \arrow[r, "Q" description] & \mathcal{D} \arrow[l, bend right, "Q_\lambda"'] \arrow[l, bend left, "Q_\rho"] \end{tikzcd} \tag{2.1}$$

which satisfy the following properties:

- $I_\lambda \dashv I \dashv I_\rho$ and $Q_\lambda \dashv Q \dashv Q_\rho$ (i.e. adjoint triples)
- I, Q_λ, Q_ρ are fully faithful
- $\ker(Q)$ coincides with the strictly full subcategory on objects of the form $I(T)$ where $T \in \mathcal{T}$.

In such a case, there are distinguished triangles

$$(Q_\lambda \circ Q)(E) \to E \to (I \circ I_\lambda)(E) \to (Q_\lambda \circ Q)(E)[1],$$
$$(I \circ I_\rho)(E) \to E \to (Q_\rho \circ Q)(E) \to (I \circ I_\rho)(E)[1]$$

which are functorial in $\mathcal{K}$. In particular, the natural transformations between these functors are given by the (co)units of the relevant adjoint pairs. As $Q_\lambda, Q, I, I_\lambda$ are left adjoints, they preserve coproducts.

3. Proof of main theorem

We start with two straightforward lemmas.

Lemma 3.1. *Consider a recollement as in Equation (2.1) where each triangulated category admits small coproducts. Assume that I preserves compact objects. If $\mathcal{T}$ and $\mathcal{D}$ are compactly generated, then so is $\mathcal{K}$. In particular, $\mathcal{K}$ is compactly generated by $I(\mathcal{T}^c)$ and $Q_\lambda(\mathcal{D}^c)$. Moreover, in such a case, if $\mathcal{T}$ and $\mathcal{D}$ are compactly generated by sets of size $\leq \beta$ for some cardinal β, then so is $\mathcal{K}$.*

Proof. By [Nee96, Theorem 5.1], the functor Q_λ preserves compact objects, since $\mathcal{D}$ is compactly generated and Q preserves small coproducts. Furthermore, Q_λ being fully faithful ensures that the unit of the adjunction $Q_\lambda \dashv Q$ is an isomorphism. Now, let $E \in \mathcal{K}$ satisfy $\mathrm{Hom}(P, E[n]) = 0$ for all $P \in \mathcal{K}^c$ and $n \in \mathbb{Z}$. We claim that $E \cong 0$, which will imply that $\mathcal{K}$ is compactly generated.

By adjunction, it follows that $\operatorname{Hom}(P', Q(E)[n]) = 0$ for all $P' \in \mathcal{D}^c$ and $n \in \mathbb{Z}$. Hence, $Q(E) = 0$ as $\mathcal{D}$ is compactly generated. So, from the distinguished triangle,

$$(I \circ I_\rho)(E) \to E \to (Q_\rho \circ Q)(E) \to (I \circ I_\rho)(E)[1],$$

we see that $(I \circ I_\rho)(E) \to E$ is an isomorphism. However, as I preserves compact objects, it follows that

$$0 = \operatorname{Hom}(I(P''), E[n]) = \operatorname{Hom}(I(P''), (I \circ I_\rho)(E)[n])$$

for all $P'' \in \mathcal{T}^c$ and $n \in \mathbb{Z}$. So, from adjunction, $\operatorname{Hom}((I_\lambda \circ I)(P''), I_\rho(E)[n]) = 0$ for all such P'' and n. However, I being fully faithful implies the counit of $I_\lambda \dashv I$ is an isomorphism, so $\operatorname{Hom}(P'', I_\rho(E)[n]) = 0$ for all such P'' and n. As $\mathcal{T}$ is compactly generated, we have that $I_\rho(E) = 0$. Consequently, the distinguished triangle above implies $E \cong 0$. This completes the proof of the first claim.

To see the last claim, let $\mathcal{B}_D$ and $\mathcal{B}_T$ respectively be sets of compacts in $\mathcal{D}$ and $\mathcal{T}$ which generate these triangulated categories and have cardinality $\leq \beta$. Let $\mathcal{B}$ consist of $P \in \mathcal{K}^c$ such that $Q(P) \in \mathcal{D}^c$ and $I_\lambda(P) \in \mathcal{T}^c$. Define $\mathcal{B}_0$ to be the subcollection of $\mathcal{B}$ choosing a representative from each equivalence class of $\mathcal{B}$ modulo objects being isomorphic in $\mathcal{K}$. Then, from the distinguished triangles,

$$(Q_\lambda \circ Q)(E) \to E \to (I \circ I_\lambda)(E) \to (Q_\lambda \circ Q)(E)[1],$$

we can check that $\mathcal{B}_0$ has cardinality $\leq \beta$. □

Lemma 3.2. *Let $F\colon \mathcal{T} \leftrightarrows \mathcal{S}\colon G$ be an adjoint pair of exact functors between triangulated categories admitting small coproducts. Assume $\mathcal{T}$ is compactly generated by a collection $\mathcal{B}$ and G commutes with small coproducts. Then G is conservative (i.e. $F(A) \cong 0 \implies A \cong 0$) if, and only if, $F(\mathcal{B})$ compactly generates $\mathcal{S}$. In such a case, if $\mathcal{T}$ is compactly generated by a set of cardinality $\leq \beta$ for some cardinal β, then $\mathcal{B}$ satisfies the same condition.*

Proof. That G commutes with small coproducts ensures that $F(\mathcal{T}^c) \subseteq \mathcal{S}^c$ (see e.g. the proof of $\implies$ in [Nee96, Theorem 5.1]). First, assume G is conservative. Let $E \in \mathcal{B}$ satisfy $\operatorname{Hom}(F(B), E[n]) = 0$ for all $B \in \mathcal{B}$ and $n \in \mathbb{Z}$. From adjunction, it follows that $\operatorname{Hom}(B, G(E)[n]) = 0$. As $\mathcal{B}$ compactly generates $\mathcal{T}$, it follows that $G(E) \cong 0$. However, G being conservative implies $E \cong 0$. So, $F(\mathcal{B})$ compactly generates $\mathcal{S}$.

Conversely, assume that $F(\mathcal{B})$ compactly generates $\mathcal{S}$. Let $E \in \mathcal{S}$ such that $G(E) \cong 0$. By adjunction, it follows that $0 = \operatorname{Hom}(F(B), E[n])$ for all $B \in \mathcal{B}$ and $n \in \mathbb{Z}$. Yet, the assumption implies $E \cong 0$. Hence, G must be conservative.

That the last claim holds follows from the proof above. □

Example 3.3. Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a quasi-affine morphism of quasi-compact quasi-separated algebraic stacks. Then $\mathbf{L}f^*\mathcal{S}$ compactly generates $D_{\mathrm{qc}}(\mathcal{Y})$ whenever $\mathcal{S} \subseteq D_{\mathrm{qc}}(\mathcal{X})^c$ does such for $D_{\mathrm{qc}}(\mathcal{X})$. Indeed, [HR17, Corollary 2.8] tells us $\mathbf{R}f_*$ is conservative as a functor on D_{qc}, and so the claim follows from Lemma 3.2.

Lemma 3.4. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a finite, flat, surjective morphism of finite presentation between quasi-compact quasi-separated algebraic stacks. If $\mathcal{Y}$ satisfies the Thomason condition, then so does $\mathcal{X}$. In particular, if $D_{\mathrm{qc}}(\mathcal{Y})$ is compactly generated by a set of cardinality $\leq \beta$ for some cardinal β, then $D_{\mathrm{qc}}(\mathcal{X})$ satisfies the same condition.*

Proof. As $f\colon \mathcal{Y} \to \mathcal{X}$ is a finite, flat, surjective, and of finite presentation, it follows that $\mathbf{R}f_*\mathcal{O}_{\mathcal{Y}} \in \operatorname{Perf}(\mathcal{X})$. Denote by $f^!$ for the right adjoint of $\mathbf{R}f_*$ on D_{qc}. In our case, [HR17, Theorem 4.14] tells us that $f^!$ preserves small coproducts and is conservative. Now, by the hypothesis, we know that there is a collection $\mathcal{B}$ (of some cardinality $\leq \beta$) which compactly generates $D_{\mathrm{qc}}(\mathcal{Y})$. So, Lemma 3.2 implies that $\mathbf{R}f_*\mathcal{B}$ compactly generates $D_{\mathrm{qc}}(\mathcal{X})$.

Next, we need to show for each quasi-compact open immersion $j\colon \mathcal{U} \to \mathcal{X}$, there is $P \in \mathrm{Perf}(\mathcal{X})$ such that $\mathrm{supp}(P) = |\mathcal{X}| \setminus |\mathcal{U}|$. Let us fix such an open immersion j. Consider the fiber square

$$\begin{array}{ccc} \mathcal{Y} \times_{\mathcal{X}} \mathcal{U} & \xrightarrow{f'} & \mathcal{U} \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{Y} & \xrightarrow[f]{} & \mathcal{X}. \end{array}$$

Base change ensures that j' is a quasi-compact open immersion. Since $\mathcal{Y}$ satisfies the Thomason condition, there is a $P \in \mathrm{Perf}(\mathcal{Y})$ with support $|\mathcal{Y}| \setminus |\mathcal{Y} \times_{\mathcal{X}} \mathcal{U}|$. As $\mathbf{R}f_*P$ is perfect, we claim that is supported on $|\mathcal{X}| \setminus |\mathcal{U}|$. Consequently, in such a case, $\mathcal{X}$ satisfies the Thomason condition.

Towards that end, observe flat base change ensures

$$\mathbf{L}j^*\mathbf{R}f_*P \cong \mathbf{R}f'_*\mathbf{L}(j')^*P \cong \mathbf{R}f'_*0 \cong 0$$

because P is supported on $|\mathcal{Y}| \setminus |\mathcal{Y} \times_{\mathcal{X}} \mathcal{U}|$. This tells us that $\mathrm{supp}(\mathbf{R}f_*P) \subseteq |\mathcal{X}| \setminus |\mathcal{U}|$. It suffices to show the reverse containment. Set $i\colon \mathcal{V} \to \mathcal{X}$ to be the quasi-compact open immersion associated to $|\mathcal{X}| \setminus \mathrm{supp}(\mathbf{R}f_*P)$. Consider the fibered square

$$\begin{array}{ccc} \mathcal{Y} \times_{\mathcal{X}} \mathcal{V} & \xrightarrow{f''} & \mathcal{V} \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{Y} & \xrightarrow[f]{} & \mathcal{X}. \end{array}$$

By base change, we know that f'' is affine. Moreover, flat base change also tells us

$$0 \cong \mathbf{L}i^*\mathbf{R}f_*P \cong \mathbf{R}f''_*\mathbf{L}(i')^*P.$$

However, $\mathbf{R}f''_*$ being conservative (see Example 3.3) ensures that $\mathbf{L}(i')^*P = 0$. So, $|\mathcal{Y} \times_{\mathcal{X}} \mathcal{V}| \subseteq |\mathcal{Y}| \setminus \mathrm{supp}(P)$, and hence, we have a string of inclusions

$$\begin{aligned} f^{-1}(|\mathcal{X}| \setminus |\mathcal{U}|) &= |\mathcal{Y}| \setminus |\mathcal{Y} \times_{\mathcal{X}} \mathcal{U}| := \mathrm{supp}(P) \\ &\subseteq |\mathcal{Y}| \setminus |\mathcal{Y} \times_{\mathcal{X}} \mathcal{V}| = f^{-1}(|\mathcal{X}| \setminus |\mathcal{V}|) = f^{-1}(\mathrm{supp}(\mathbf{R}f_*P)). \end{aligned}$$

Consequently, f being surjective promises that $|\mathcal{X}| \setminus |\mathcal{U}| \subseteq \mathrm{supp}(\mathbf{R}f_*P)$. ◻

Lemma 3.5. *Let $i\colon \mathcal{Z} \to \mathcal{X}$ be a closed immersion to a quasi-compact quasi-separated algebraic stack. If $\mathcal{X}$ satisfies the Thomason condition, then so does $\mathcal{Z}$. In particular, if $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated by a set of cardinality $\leq \beta$ for some cardinal β, then $D_{\mathrm{qc}}(\mathcal{Z})$ satisfies the same condition.*

Proof. That $D_{\mathrm{qc}}(\mathcal{Z})$ is compactly generated follows from Example 3.3. So, we show for every quasi-compact open immersion $\mathcal{U} \to \mathcal{Z}$, there is a $P \in \mathrm{Perf}(\mathcal{Z})$ such that $\mathrm{supp}(P) = |\mathcal{Z}| \setminus |\mathcal{U}|$. Let $W := |\mathcal{Z}| \setminus |\mathcal{U}|$. As $\mathcal{X}$ satisfies the Thomason condition, there is a $P \in \mathrm{Perf}(X)$ such that $\mathrm{supp}(P) = W$. It is straightforward to check that $\mathbf{L}i^*P \in \mathrm{Perf}(\mathcal{Z})$. Moreover, by [HR17, Lemma 4.8(2)], we know that

$$\mathrm{supp}(\mathbf{L}i^*P) = i^{-1}(\mathrm{supp}(P)) = i^{-1}(W) = W.$$

This completes the proof. ◻

Proposition 3.6. *Let $\mathcal{X}$ be a regular Noetherian algebraic stack with quasi-finite diagonal. Then $\mathcal{X}$ satisfies the 1-Thomason condition.*

Proof. By [Ryd11, Theorem 7.2], there is an étale surjective morphism $p\colon \mathcal{Y} \to \mathcal{X}$ which is representable by algebraic spaces and of finite presentation, as well as a finite, flat, surjective morphism $v\colon U \to \mathcal{Y}$ which is of finite presentation from a quasi-affine scheme. Moreover, [Ryd11, Proposition 4.4] gives us finite sequence of quasi-compact open immersions

$$\emptyset =: \mathcal{X}_0 \xrightarrow{j_0} \mathcal{X}_1 \xrightarrow{j_1} \cdots \xrightarrow{j_{n-2}} \mathcal{X}_{n-1} \xrightarrow{j_{n-1}} \mathcal{X}_n =: \mathcal{X}$$

such that the base change of f to $|\mathcal{X}_c| \setminus |\mathcal{X}_{c-1}|$ (when given any closed substack structure) is étale finite of constant rank. Observe, from Lemma 3.4, that $\mathcal{Y}$ satisfies the 1-Thomason condition because v is a finite, flat, surjective morphism of finite presentation whose source satisfies the 1-Thomason condition.

By inducting on n, we may restrict to the setting where f is finite after base change along a closed immersion $t\colon \mathcal{Z} \to \mathcal{X}$ and the open substack $\mathcal{U} := |\mathcal{X}| \setminus t(|\mathcal{Z}|)$ is 1-Thomason. Set $Z = t(|\mathcal{Z}|)$ and $j\colon \mathcal{U} \to \mathcal{X}$ be the associated open immersion. There is a recollement

$$D_{\mathrm{qc},Z}(\mathcal{X}) \underset{i^!}{\overset{i^*}{\leftleftarrows}} \xrightarrow{i_*} D_{\mathrm{qc}}(\mathcal{X}) \underset{\mathbf{R}j_*}{\overset{j_!}{\leftleftarrows}} \xrightarrow{\mathbf{L}j^*} D_{\mathrm{qc}}(\mathcal{U}).$$

Using Lemma 3.1, it suffices to show that i_* preserves compact objects and $D_{\mathrm{qc},Z}(\mathcal{X})$ is singly compactly generated. Towards that end, let $f'\colon \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$ be the base change of f along t. In this situation, f' is a finite, étale, surjective morphism which is representable by algebraic spaces. By Lemma 3.5, $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ is 1-Thomason because $\mathcal{Y}$ is such, and so once again, Lemma 3.4 ensures $\mathcal{Z}$ satisfies the 1-Thomason condition because f' is finite, étale, surjective, and of finite presentation.

As $\mathcal{X}$ is regular and Noetherian, it follows that $\operatorname{Perf}(\mathcal{X}) = D^b_{\mathrm{coh}}(\mathcal{X})$ (see e.g. [DLMP25, Theorem 3.7]). This ensures that $\mathbf{R}t_* D^b_{\mathrm{coh}}(\mathcal{Z}) \subseteq \operatorname{Perf}(\mathcal{X})$. Hence, by [HR17, Theorem 4.14(4)], the right adjoint $t^\times$ of $\mathbf{R}t_*$ on D_{qc} preserves small coproducts. We claim that the adjunction $\mathbf{R}t_* \dashv t^\times$ restricts to an adjoint pair between $D_{\mathrm{qc}}(\mathcal{Z})$ and $D_{\mathrm{qc},Z}(\mathcal{X})$. To do so, we only need to check that $\mathbf{R}t_* D_{\mathrm{qc}}(\mathcal{Z}) \subseteq D_{\mathrm{qc},Z}(\mathcal{X})$. However, being that $D_{\mathrm{qc}}(\mathcal{Z})$ is compactly generated, every object of $D_{\mathrm{qc}}(\mathcal{Z})$ is a homotopy colimit of iterated extensions of small coproducts of $\operatorname{Perf}(\mathcal{Z})$ (see [Sta25, Tag 09SN]). So, this can be checked by showing $\mathbf{R}t_* \operatorname{Perf}(\mathcal{Z}) \subseteq D_{\mathrm{qc},Z}(\mathcal{X})$. Yet, this is known as $\mathbf{R}t_* \operatorname{Perf}(\mathcal{Z}) \subseteq D^b_{\mathrm{coh},Z}(\mathcal{X})$ (see e.g. [DLMP25, Lemma 4.6]).

Next, after showing $\mathbf{R}t_* \dashv t^\times$ restricts to an adjoint pair between $D_{\mathrm{qc}}(\mathcal{Z})$ and $D_{\mathrm{qc},Z}(\mathcal{X})$, we claim $t^\times$ (restricted) is conservative. Let $E \in D_{\mathrm{qc},Z}(\mathcal{X})$ satisfy $t^\times E \cong 0$. By [HR17, Theorem 4.14(2)], it follows that

$$0 \cong \mathbf{R}t_* t^\times E \cong \mathbf{R}\mathcal{H}om(\mathbf{R}t_*\mathcal{O}_{\mathcal{Z}}, E).$$

However, $\mathbf{R}t_*\mathcal{O}_{\mathcal{Z}}$ is perfect with support Z, and so, [HR17, Lemma 4.9] ensures that $E \cong 0$. Hence, Lemma 3.2 implies $\mathbf{R}t_* D_{\mathrm{qc}}(\mathcal{Z})^c$ compactly generates $D_{\mathrm{qc},Z}(\mathcal{X})$.

We are left to show that the compacts of $D_{\mathrm{qc},Z}(\mathcal{X})$ are also compact in $D_{\mathrm{qc}}(\mathcal{X})$. Luckily, this is easier. Namely, [HR17, Theorem 4.14(4)] tells us $t^\times\colon D_{\mathrm{qc}}(\mathcal{X})$ preserves small coproducts. So, e.g. the proof of $\implies$ in [Nee96, Theorem 5.1], ensures that $\mathbf{R}t_* D_{\mathrm{qc}}(\mathcal{Z})^c \subseteq D_{\mathrm{qc}}(\mathcal{X})^c$. Consequently, Lemma 3.1 tells us we finished the proof. □

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